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On the general formulation of meson pair theory

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Introduction

Meson theory offers in the main the following three explanations of the experimental fact that at least in the singlet state the proton-proton and proton-neutron interactions are nearly equal:

1) The first alternative is a symmetrical meson theory of the type proposed by Kemmer [17]. In such a theory the interaction between the nucleons is due in first order of approximation to the virtual transfer of single mesons from one nucleon to another, and by means of a symmetrical interaction with neutral and charged meson fields the proton-proton and proton-neutron interactions are made equal. The transfer of single mesons is described mathematically by assuming the interaction term in the Hamiltonian to be linear in the meson field quantities, in close analogy to the electromagnetic case.

2) The second possibility consists in assuming an interaction term in the Hamiltonian, which is linear in the meson field quantities, but which involves only neutral mesons. Since the virtual transfer of a neutral meson does not alter the charge of the nucleons, the resulting interaction must be independent of the nucleonic charge, as is desired. Such a theory has been proposed and worked out by Bethe [1]. A theory of this type has the disadvantage of operating

only with neutral mesons, the existence of which is rather uncertain, whereas no rôle is assigned to the charged mesons, which have without doubt been observed both in cosmic and cyclotron rays. Moreover, neutral mesons alone cannot account for the β-radioactivity and the magnetic moments of the nucleons.

3) The third possibility is the assumption that the interaction between the nucleons is produced in first order of approximation by the virtual transfer from one nucleon to another of pairs of one positive and one negative meson. Since such a transfer does not alter the charge of the nucleons, the desired equality between proton-proton and proton-neutron interaction is automatically ensured. The characteristic feature of such a charged meson pair theory is that the charge of the nucleons and the charge of the mesons are conserved separately, i. e. that a continuity condition holds for each of these charges. In the quantized theory this means that the total charge of the nucleons and the total charge of the mesons are separately constants of the motion, i. e. they both commute with the total Hamiltonian. This property can be attained by assuming the interaction term in the Hamiltonian to be bilinear both in the meson field and in the nucleonic field. This means, that the meson field equations are homogeneous with respect to the meson field quantities with an interaction term which is bilinear in the nucleonic wave function, whereas the wave equation for the nucleons is homogeneous with respect to the wave function of the nucleons with an interaction term which is bilinear in the meson field quantities. In the case where the field particles are heavy electrons (spin $^1/_2$ particles) all the LORENTZ invariant pair interactions have been listed and discussed by MARSHAK [25]. For mesons in the proper sense (spin 0 and 1 particles) various cases have been investigated for scalar mesons by WENTZEL [32], JAUCH & LOPES [16] and PAULI & HU [27], and for vector mesons by KLEIN [20], PAULI & HU [27] and BRULIN & HJALMARS [14], [4], [5]. (Pair theories involving both charged and neutral mesons have been discussed by NOMA [26] and BRULIN & HJALMARS [6]).

All the methods 1)—3) are capable in principle of explaining most of the other experimental facts about nuclear forces, although the numerical agreement is in most cases rather unsatisfactory. On the other hand, all three methods are afflicted with the divergency difficulties at present inherent in any quantum field theory, and they all require a cut-off procedure, although not always in the first approximation. However, we may say that they are all still open possibilities.

If, however, we reject the neutral meson, as not being experimentally established, we must conclude that the main part of the singlet potential is due to forces of the pair theory type. Of course we can in addition have forces of the type of the ordinary charged meson theory, but they must make a very small contribution to the singlet state, since they give different results for proton-proton and proton-neutron interaction. In fact such a combination of ordinary and pair theory forces may be afforded by KLEIN's 5-dimensional theory [21].

The purpose of the present paper is to formulate the most general pair interaction between nucleons and charged scalar and vector mesons. As was pointed out in an earlier note [15], a natural and the most convenient method of developing the meson pair theory is to make use of the generalized DIRAC equation for the meson. For this reason we make in Chapter I a review of the theory of generalized DIRAC equations and derive the conditions for the LORENTZ invariance and Hermiticity of the pair interaction term and for the conservation of the mesonic and nucleonic charge. In Chapters II, III and IV we apply the general theory of Chapter I to the nucleons and to the scalar and vector mesons. Although not revealing anything essentially new, this gives a convenient representation of KEMMER's β_k:s, in which the general matrix element is expressed as a function only of KRONECKER symbols of suitably chosen indices. This representation will presumably be useful not only in the pair theory. In Chapter V the pair theory is formulated and the most general pair interactions, expressed in the β_k:s of the mesons and the γ_k:s of the nucleons, are listed. In Chapter VI we show that it is possible to perform a charge symmetrical quantization of the meson field, in proving that with the given representation of the β_k:s, the charge conjugate wave function is identical with the transposed adjoint wave function. In Chapters VII and VIII we eliminate the redundant components and express the Hamiltonian as a function of the non-redundant components and their spatial derivatives only. In the Appendix, finally, we give a brief discussion of the alternative method of developing the pair theory by means of the LAGRANGE formalism.

The implications of the meson pair theory for the nuclear two-body problem will be discussed in some detail by BRULIN in two papers which are to be printed in this journal [7].

I am indebted to professor OSKAR KLEIN for suggesting the problem treated in this thesis and for his helpful interest and valuable advice during the work. I want especially to express my gratitude for his kindness of putting at my disposal some unpublished manuscripts of his, which constitute the starting point of Chapters III and IV.

My thanks are further due to my colleague fil. lic. OLOF BRULIN, with whom I have had the most intimate collaboration in preparing this paper.

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CHAPTER I

A review of the theory of general Dirac equations

The possibility of writing the equations of other elementary particles than the electron as a generalized DIRAC equation,

$$\left(\sum_{k=1}^{4} \Gamma_k p_k - \lambda\right) \psi = 0, \tag{1}$$

where:

$$p_k = -i\hbar \frac{\partial}{\partial x_k} \qquad (k = 1, 2, 3), \tag{2}$$

$$p_4 = -i\hbar \frac{\partial}{\partial x_4} = -\frac{\hbar}{c}\frac{\partial}{\partial t} = \frac{i}{c} E, \tag{3}$$

was first investigated in various cases by MAJORANA [23] in 1932, by DE BROGLIE [2], [3] and pupils, e. g. PETIAU [28], [29], in 1934—36 and by DUFFIN [9] and KEMMER [18] in 1938. A general theory, in which these cases are contained, was developed by KLEIN [19] in 1936. In this chapter we will give a survey of KLEIN's theory, paying special attention to the term λ of equ. (1), which is of particular interest in meson pair theories.

We investigate the conditions for the invariance of equ. (1) under a LORENTZ transformation, by which any 4-vector or tensor index, e. g. p_k, is transformed according to the equation:

$$p'_k = \sum_{l=1}^{4} a_{kl}\, p_l \tag{4}$$

with the condition of orthogonality

$$\sum_{k=1}^{4} a_{kl}\, a_{km} = \sum_{k=1}^{4} a_{lk}\, a_{mk} = \delta_{lm}. \tag{5}$$

The Γ_k:s are supposed to be coefficients which are *not* transformed, but the wave function ψ is transformed according to the equation

$$\psi' = Q\,\psi, \tag{6}$$

where Q is a transformation operator, corresponding to the LORENTZ transformation of equ. (4). The term λ may in general be assumed to contain both coefficients like the Γ_k:s, which are not transformed, and tensors, $T_{kl\ldots}$, which are transformed by equ. (4). Let λ' be the same function of the Γ_k:s and $T'_{kl\ldots}$:s as λ is of the Γ_k:s and $T_{kl\ldots}$:s. Now the transformed equation must be:

$$\left(\sum_{k=1}^{4} \Gamma_k\, p'_k - \lambda'\right)\psi' = 0,$$

and this equation must give equ. (1) if we insert eqs. (4) and (6). We thus get:

$$\left(\sum_{k=1}^{4}\sum_{l=1}^{4} \Gamma_k\, Q\, a_{kl}\, p_l - \lambda'\, Q\right)\psi = 0.$$

If we multiply this equation from the left by Q^{-1}, we get:

$$\left(\sum_{k=1}^{4}\sum_{l=1}^{4} Q^{-1}\,\Gamma_k\, Q\, a_{kl}\, p_l - Q^{-1}\lambda'\, Q\right)\psi = 0.$$

This equation is evidently identical with equ. (1) if

$$\sum_{k=1}^{4} Q^{-1}\,\Gamma_k\, Q\, a_{kl} = \Gamma_l \tag{7}$$

and

$$Q^{-1}\lambda'\, Q = \lambda. \tag{8}$$

Multiplying equ. (7) by a_{ml}, summing over l and using equ. (5) we get the well-known result:

$$Q^{-1}\,\Gamma_m Q = \sum_{l=1}^{4} a_{ml}\,\Gamma_l\,. \tag{9}$$

If Γ_l is considered as a 4-vector, the canonical transformation $Q^{-1}\cdots Q$ must thus be identical with the LORENTZ transformation.

KLEIN now considers the identity of equ. (9) in n dimensions, when the transformations are restricted to being infinitesimal. From such transformations all finite, proper LORENTZ transformations can be built up. For the infinitesimal transformations we can put

$$a_{kl} = \delta_{kl} + \varepsilon_{kl}\,, \tag{10}$$

and equ. (5) then gives:

$$\varepsilon_{kl} = -\,\varepsilon_{lk}\,. \tag{11}$$

The identity of equ. (9) can now be written:

$$Q^{-1}\,\Gamma_l\,Q = \Gamma_l + \sum_{k=1}^{n} \varepsilon_{lk}\,\Gamma_k\,. \tag{12}$$

If we expand Q and Q^{-1} in terms of the ε_{kl}:s, we get in the first approximation:

$$Q = 1 + \tfrac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \varepsilon_{kl}\,q_{kl} \tag{13}$$

and

$$Q^{-1} = 1 - \tfrac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \varepsilon_{kl}\,q_{kl}\,. \tag{14}$$

Since the ε_{kl}:s are antisymmetrical in k and l, we can without lack of generality assume that the operators q_{kl} are so too, i. e.

$$q_{kl} = -\,q_{lk}\,. \tag{15}$$

As is well-known, equ. (12) gives the following commutation rules for the Γ_k:s and the q_{kl}:s:

$$[q_{kl}\,,\ \Gamma_m] = \delta_{lm}\,\Gamma_k - \delta_{km}\,\Gamma_l \tag{16}$$

and

$$[q_{kl}\,,\ q_{rs}] = \delta_{lr}\,q_{ks} - \delta_{kr}\,q_{ls} + \delta_{ls}\,q_{rk} - \delta_{ks}\,q_{rl}\,. \tag{17}$$

KLEIN shows that if we consider the n-dimensional space as a subspace of a $(n+1)$-dimensional space and put

$$\Gamma_m = i\,q_{m,\,n+1} \qquad (m = 1\,.\,.\,.\,n), \tag{18}$$

these Γ_m:s satisfy equ. (16) for $k, l, m = 1\,.\,.\,.\,n$, and give

$$q_{kl} = [\Gamma_k\,,\ \Gamma_l] \qquad (k, l = 1\,.\,.\,.\,n). \tag{19}$$

Now eqs. (16) and (19) give:

$$[[\Gamma_k, \Gamma_l], \Gamma_m] = \delta_{lm}\,\Gamma_k - \delta_{km}\,\Gamma_l \quad (k, l, m = 1 \ldots n). \tag{20}$$

Alternatively equ. (20) can be taken as the starting point, and equ. (19) as a definition of the q_{kl}:s, since, as is easily seen, eqs. (16) and (17) follow from eqs. (19) and (20). We see that if we have found the q_{kl}:s for a 5-dimensional space, fulfilling equ. (17), the quantities $\Gamma_m = i\,q_{m5}$ $(m = 1, 2, 3, 4)$ are possible coefficients in equ. (1), giving the transformation operator

$$Q = 1 + \tfrac{1}{2}\sum_{k=1}^{4}\sum_{l=1}^{4}\varepsilon_{kl}[\Gamma_k, \Gamma_l]. \tag{21}$$

Of course eqs. (19) and (20) only define a special case of the general case defined by eqs. (16) and (17), but it may well be that in future it will prove to contain all cases of physical interest, since in fact it has so far proved to do so. A theory, which conforms to equ. (20) but not to equ. (19), is for example ERIKSSON's [10], as has been shown by BRUNO [8].

As has been pointed out by KLEIN in the paper from 1936, and as has later been more fully discussed by KRAMERS, BELINFANTE & LUBAŃSKI [22], the irreducible sets of Γ_k:s defined by equ. (20) can be found by reduction of the sums of commuting sets of ordinary DIRAC matrices divided by 2.

In the following treatment we shall only consider such representations of equ. (20) in which all Γ_k:s are Hermitian i. e.

$$\Gamma_k^* = \Gamma_k,$$

if we denote the Hermitian conjugate of an operator by a star. Then the q_{kl}:s are anti-Hermitian, since:

$$q_{kl}^* = (\Gamma_k\,\Gamma_l - \Gamma_l\,\Gamma_k)^* = \Gamma_l\,\Gamma_k - \Gamma_k\,\Gamma_l = -\,q_{kl}.$$

Let us now consider the adjoint equation

$$\tilde{\psi}\left(\sum_{k=1}^{4}\Gamma_k\,p_k - \lambda\right) = 0, \tag{22}$$

where

$$p_k = i\hbar\,\frac{\partial}{\partial x_k} \qquad (k = 1, 2, 3) \tag{23}$$

and

$$p_4 = i\hbar\,\frac{\partial}{\partial x_4} = \frac{\hbar}{c}\,\frac{\partial}{\partial t} = \frac{i}{c}\,E. \tag{24}$$

If we insert equ. (4) and the assumption

$$\tilde{\psi}' = \tilde{\psi}\,Q^{-1} \tag{25}$$

in the transformed equation

$$\tilde{\psi}'\left(\sum_{k=1}^{4} \Gamma_k p_k' - \lambda'\right) = 0$$

and multiply by Q from the right, we get

$$\tilde{\psi}\left(\sum_{k=1}^{4}\sum_{l=1}^{4} Q^{-1}\,\Gamma_k\,Q\,a_{kl}\,p_l - Q^{-1}\,\lambda'\,Q\right) = 0.$$

This equation is evidently identical with equ. (22) if eqs. (7) and (8) are fulfilled. The adjoint equation is thus invariant, if this is the case for the original equation.

We will now proceed to determine the relation between ψ and $\tilde{\psi}$, following a method, wellknown from the ordinary DIRAC equation, and generalized to the meson case by PETIAU [29]. We introduce the Hermitian operator

$$p_0 = -i\,p_4 = \frac{E}{c}, \tag{26}$$

and an Hermitian operator Γ fulfilling the following relations

$$\Gamma^2 = 1, \tag{27}$$

$$[\Gamma, \Gamma_4] = 0, \tag{28}$$

$$\Gamma\,\Gamma_k + \Gamma_k\,\Gamma = 0 \quad (k = 1, 2, 3). \tag{29}$$

We now form the Hermitian operators:

$$a_0 = 2\,\Gamma\,\Gamma_4 \tag{30}$$

and

$$a_k = -2\,i\,\Gamma\,\Gamma_k \quad (k = 1, 2, 3). \tag{31}$$

Multiplying equ. (1) from the left by $-2\,i\,\Gamma$ we get:

$$\left(\sum_{k=0}^{3} a_k\,p_k + 2\,i\,\Gamma\lambda\right)\psi = 0. \tag{32}$$

If we form the Hermitian conjugate of this equation we get:

$$\psi^*\left(\sum_{k=0}^{3} a_k\,p_k - 2\,i\,(\Gamma\lambda)^*\right) = 0.$$

If we now again insert eqs. (30) and (31) and divide by $-2\,i$ we have:

$$(\psi^*\,\Gamma)\left(\sum_{k=1}^{4} \Gamma_k\,p_k + \Gamma\,(\Gamma\lambda)^*\right) = 0.$$

Using equ. (22) we can put

$$\tilde{\psi} = \psi^* \Gamma \tag{33}$$

if

$$\Gamma (\Gamma \lambda)^* = - \lambda$$

or

$$(\Gamma \lambda)^* = - \Gamma \lambda, \tag{34}$$

i. e. if $\Gamma \lambda$ is anti-Hermitian. The assumption of equ. (33) can be made only if $\psi^* \Gamma$ is transformed by equ. (25), i. e. if

$$\psi^{*\prime} \Gamma = \psi^* \Gamma Q^{-1}$$

or

$$\psi^{*\prime} = \psi^* \Gamma Q^{-1} \Gamma. \tag{35}$$

On the other hand we have from equ. (6):

$$\psi^{*\prime} = \psi^* Q^*. \tag{36}$$

Comparing eqs. (35) and (36) wee see that if

$$Q^* = \Gamma Q^{-1} \Gamma, \tag{37}$$

then the assumption of equ. (33) is justified. Now equ. (37) can easily be verified using eqs. (13), (14), (19), (27), (28), (29) and

$$\overset{*}{\varepsilon}_{kl} = \varepsilon_{kl}, \; \overset{*}{\varepsilon}_{k4} = - \varepsilon_{k4} \quad (k, l = 1, 2, 3).$$

We are now also able to give the Q, corresponding to a spatial reflexion,

$$x_k \to - x_k \; (k = 1, 2, 3), \; x_4 \to x_4. \tag{38}$$

In fact, as is easily seen, $Q = \pm \Gamma$ satisfy in this case eqs. (9) and (37).

We will now investigate how the quantities $\tilde{\psi} \Gamma_k \Gamma_l \cdots \Gamma_n \psi$ are transformed under a LORENTZ transformation provided that the Γ_k:s are operators giving the same result for this expression, no matter whether they are applied to the right on the ψ or to the left on the $\tilde{\psi}$. According to eqs. (6), (25) and (9) we have:

$$\tilde{\psi}' \Gamma_k \Gamma_l \cdots \Gamma_n \psi' = \tilde{\psi} Q^{-1} \Gamma_k Q Q^{-1} \Gamma_l Q \cdots Q^{-1} \Gamma_n Q \psi =$$

$$= \sum_{r=1}^{4} \sum_{s=1}^{4} \cdots \sum_{t=1}^{4} a_{kr} a_{ls} \cdots a_{nt} \tilde{\psi} \Gamma_r \Gamma_s \cdots \Gamma_t \psi. \tag{39}$$

The quantity $\tilde{\psi} \Gamma_k \Gamma_l \cdots \Gamma_n \psi$ is thus transformed like a tensor. $\tilde{\psi} \psi$ especially is an invariant and $\tilde{\psi} \Gamma_k \psi$ is a 4-vector. Consider the 4-dimensional divergence of this vector:

$$\sum_{k=1}^{4} \frac{\partial}{\partial x_k} \tilde{\psi} \Gamma_k \psi = \sum_{k=1}^{4} \frac{\partial \tilde{\psi}}{\partial x_k} \Gamma_k \psi + \sum_{k=1}^{4} \tilde{\psi} \Gamma_k \frac{\partial \psi}{\partial x_k} =$$

$$= \frac{1}{i \hbar} \left\{ \sum_{k=1}^{4} (\tilde{\psi} p_k) \Gamma_k \psi - \sum_{k=1}^{4} \tilde{\psi} \Gamma_k (p_k \psi) \right\}. \tag{40}$$

If we multiply equ. (1) by $\tilde{\psi}$ from the left and equ. (22) by ψ from the right and subtract, we obtain:

$$\sum_{k=1}^{4} \tilde{\psi}\, \Gamma_k (p_k \psi) - \sum_{k=1}^{4} (\tilde{\psi}\, p_k)\, \Gamma_k \psi = \tilde{\psi}\, (\lambda \psi) - (\tilde{\psi} \lambda)\, \psi. \tag{41}$$

Under the condition that

$$\tilde{\psi}\, (\lambda \psi) = (\tilde{\psi} \lambda)\, \psi, \tag{42}$$

it follows from (40) and (41):

$$\sum_{k=1}^{4} \frac{\partial}{\partial x_k} \tilde{\psi}\, \Gamma_k \psi = 0. \tag{43}$$

$\tilde{\psi}\, \Gamma_k \psi$ can thus be considered as proportional to the current 4-vector, so that the quantity

$$\tilde{\psi}\, \Gamma_4 \psi = \psi^* \Gamma\, \Gamma_4 \psi \tag{44}$$

is proportional to the density of probability.

In our treatment of the meson pair theory we shall assume λ to contain a term which is responsible for the interaction between the nucleons and the meson field. Consequently, eqs. (8), (34) and (42) are of special importance in the pair theory, since they express the conditions which are to be imposed on the pair interaction term in order to ensure relativistic invariance, Hermiticity and conservation of charge respectively. In Chapter V we shall see how these conditions limit the number of possible interactions.

CHAPTER II

The ordinary Dirac equation

In the present treatment of the meson pair theory we will assume that the nucleons obey the ordinary DIRAC equation, and that the meson fields obey the generalized DIRAC equations, in which the wave function is a vector or an antisymmetrical tensor. In the present chapter we shall apply KLEIN's general theory, sketched in the preceding chapter, to the first of these cases. It is easy to see that if we put

$$\Gamma_k = \tfrac{1}{2} \gamma_k, \tag{45}$$

where the γ_k:s are ordinary DIRAC matrices, fulfilling the conditions

$$\gamma_k \gamma_l + \gamma_l \gamma_k = 2\, \delta_{kl} \quad (k, l = 1, 2, 3, 4), \tag{46}$$

then the Γ_k:s satisfy equ. (20) and conform thus to the general theory. In this case

$$\Gamma = \gamma_4 \tag{47}$$

satisfy the conditions of eqs. (27), (28) and (29). We have then by equ. (33):

$$\tilde{\psi} = \psi^* \gamma_4 . \tag{48}$$

Eqs. (30) and (31) give

$$a_0 = 1 \tag{49}$$

and

$$a_k = -\, i \gamma_4 \gamma_k \quad (k = 1, 2, 3), \tag{50}$$

and if we put

$$a_m = \gamma_4 , \tag{51}$$

our a_k:s are the ordinary a_k:s of DIRAC, satisfying

$$a_k a_l + a_l a_k = 2\, \delta_{kl} \quad (k, l = 1, 2, 3, m). \tag{52}$$

Now put

$$\lambda = \tfrac{1}{2} (\vartheta + \pi), \tag{53}$$

where ϑ is a term, which, like the coefficients γ_k, does not change under a LORENTZ transformation, and where π is a term containing both coefficients like the γ_k:s and tensors. We have according to equ. (8):

$$Q^{-1} \vartheta Q = \vartheta \tag{54}$$

and

$$Q^{-1} \pi' Q = \pi . \tag{55}$$

The term π, which can be used as the interaction term in the meson pair theory, will be discussed in Chapter V.

Since every 4-rowed square matrix can be expressed as a linear polynomial of the four γ_k:s and their products, ϑ must be such a sum. Since the transformation Q is according to equ. (9) identical with a LORENTZ transformation with the γ_k:s considered as vector components, equ. (54) means that ϑ must have the form of an invariant in the γ_k:s. As is well-known, the most general form of such an invariant and thus of ϑ is a constant times the unit matrix.

If in conformity with equ. (34) we put

$$\vartheta = -\, i M c \tag{56}$$

we get by equ. (1):

$$\left(\sum_{k=1}^{4} \gamma_k p_k + i M c - \pi \right) \psi = 0 \tag{57}$$

and by equ. (32):

$$\left(\frac{E}{c} + \sum_{k=1}^{3} a_k p_k + a_m M c + i\, a_m \pi \right) \psi = 0, \tag{58}$$

in conformity with DIRAC's formulation if we put $\pi = 0$.

For later use we will now list the well-known expressions for the tensor components

$$\tilde{\psi}\psi,\ \tilde{\psi}\gamma_k\psi,\ \tilde{\psi}\gamma_k\gamma_l\psi,\ \tilde{\psi}\gamma_k\gamma_l\gamma_m\psi,\ \tilde{\psi}\gamma_k\gamma_l\gamma_m\gamma_n\psi, \tag{59}$$

where all indices k, l, m and n are different, in terms of the two commuting systems of PAULI matrices $\sigma_1, \sigma_2, \sigma_3$ and $\varrho_1, \varrho_2, \varrho_3$, both fulfilling equations of the type

$$\sigma_k\sigma_l + \sigma_l\sigma_k = 2\,\delta_{kl} \tag{60}$$

$$\sigma_1\sigma_2 = i\,\sigma_3 \qquad \text{(cyclic permutations)} \tag{61}$$

and according to DIRAC being related to the α_k:s by means of the equations:

$$\alpha_k = \varrho_1\sigma_k \qquad (k = 1,\,2,\,3) \tag{62}$$

and

$$\alpha_m = \varrho_3\,. \tag{63}$$

We have:

$$\tilde{\psi}\psi = \psi^*\gamma_4\psi = \psi^*\varrho_3\psi, \tag{64}$$

$$\tilde{\psi}\gamma_k\psi = \psi^*\gamma_4\gamma_k\psi = i\,\psi^*\varrho_1\sigma_k\psi, \tag{65}$$

$$\tilde{\psi}\gamma_4\psi = \psi^*\gamma_4\gamma_4\psi = \psi^*\psi, \tag{66}$$

$$\tilde{\psi}\gamma_k\gamma_l\psi = \psi^*\gamma_4\gamma_k\gamma_l\psi = i\,\psi^*\varrho_3\sigma_{kl}\psi, \tag{67}$$

$$\tilde{\psi}\gamma_4\gamma_k\psi = \psi^*\gamma_4\gamma_4\gamma_k\psi = -\,\psi^*\varrho_2\sigma_k\psi, \tag{68}$$

$$\tilde{\psi}\gamma_1\gamma_2\gamma_3\psi = \psi^*\gamma_4\gamma_1\gamma_2\gamma_3\psi = -\,\psi^*\varrho_1\psi, \tag{69}$$

$$\tilde{\psi}\gamma_4\gamma_k\gamma_l\psi = \psi^*\gamma_4\gamma_4\gamma_k\gamma_l\psi = i\,\psi^*\sigma_{kl}\psi, \tag{70}$$

$$\tilde{\psi}\gamma_1\gamma_2\gamma_3\gamma_4\psi = \psi^*\gamma_4\gamma_1\gamma_2\gamma_3\gamma_4\psi = i\,\psi^*\varrho_2\psi, \tag{71}$$

where $k, l = 1, 2, 3$ and $\sigma_{kl} = \pm\,\sigma_m$, m being the number 1, 2, 3, different from k and l, and the + sign being used if k, l and m are 1, 2, 3 or cyclic permutations, the — sign if this is not the case.

CHAPTER III

The scalar and pseudoscalar meson theory

The matrices in this case have been examined in detail by DUFFIN [9] and KEMMER [18]. In this chapter we shall, following a method suggested by KLEIN, derive a very natural representation of these matrices, and give a formula for the general matrix element, only involving KRONECKER symbols of suitably chosen indices. Of course we will not in this way derive anything which is not essentially contained in the above mentioned papers by DUFFIN and KEMMER, but our general formula for the matrix element will prove very convenient for the calculation of all sorts of products of the matrices.

Suppose that the wave function of the generalized DIRAC equation (1) is transformed like a 5-dimensional vector Ψ_k $(k = 1, 2, 3, 4, 5)$. This means of course that for transformations restricted to the 4-dimensional subspace $(\varepsilon_{k5} = 0)$

$$\Psi_1, \Psi_2, \Psi_3, \Psi_4$$

transform like a 4-vector and Ψ_5 like an invariant. We have:

$$\Psi'_k = \Psi_k + \sum_{l=1}^{5} \varepsilon_{kl} \Psi_l . \tag{72}$$

On the other hand, denoting all matrices belonging to this case with a bar, equ. (6) gives

$$\Psi'_k = \left[\left(1 + \tfrac{1}{2} \sum_{r=1}^{5} \sum_{s=1}^{5} \varepsilon_{rs} \bar{q}_{rs} \right) \Psi \right]_k . \tag{73}$$

The equations (72) and (73) are supposed to be identical. Equ. (72) can be written:

$$\Psi'_k = \Psi_k + \tfrac{1}{2} \sum_{r=1}^{5} \sum_{s=1}^{5} \varepsilon_{rs} (\delta_{kr} \Psi_s - \delta_{ks} \Psi_r) \tag{74}$$

and equ. (73) implies:

$$\Psi'_k = \Psi_k + \tfrac{1}{2} \sum_{r=1}^{5} \sum_{s=1}^{5} \varepsilon_{rs} (\bar{q}_{rs} \Psi)_k . \tag{75}$$

Identification of eqs. (74) and (75) gives

$$(\bar{q}_{rs} \Psi)_k = \delta_{kr} \Psi_s - \delta_{ks} \Psi_r . \tag{76}$$

Equ. (76) has been derived in an unpublished manuscript by prof. KLEIN, who has kindly communicated it to the author.

We will here proceed to derive the expression for the matrix element of $\bar{q}_{kl}$. For this purpose we have only to write equ. (76) in the form

$$(\bar{q}_{kl} \Psi)_n = \sum_{n'=1}^{5} (\delta_{nk} \delta_{n'l} - \delta_{nl} \delta_{n'k}) \Psi_{n'} . \tag{77}$$

If we consider Ψ as a column-matrix of the form

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \end{pmatrix} \tag{78}$$

and $\bar{q}_{rs}$ as a 5-rowed square matrix operating to the right on the Ψ, equ. (77) can clearly be interpreted so that the matrix elements of $\bar{q}_{kl}$ are:

$$(n\,|\,\bar{q}_{kl}\,|\,n') = \delta_{nk}\,\delta_{n'l} - \delta_{nl}\,\delta_{n'k}. \tag{79}$$

If $\bar{\beta}_k$ are the 5-rowed Γ_k:s in this case, we have by equ. (18):

$$(n\,|\,\bar{\beta}_k\,|\,n') = i\,(\delta_{nk}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'k}). \tag{80}$$

The $\bar{\beta}_k$:s so defined are Hermitian, since

$$(n\,|\,\bar{\beta}_k\,|\,n')^* = i\,(\delta_{n5}\,\delta_{n'k} - \delta_{nk}\,\delta_{n'5}) = (n'\,|\,\bar{\beta}_k\,|\,n). \tag{81}$$

The form of the matrices is the following:

$$\bar{\beta}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \bar{\beta}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\beta}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \end{pmatrix}, \qquad \bar{\beta}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & -i & 0 \end{pmatrix}. \tag{82}$$

We wish to draw attention to the fact that all the elements of the Hermitian matrices are purely imaginary, which means that the matrices only change their sign when they are transposed, i. e. rows and columns interchanged, a property of which we will make use in our consideration of the charge conjugation in the meson pair theory.

To prove that the matrices obtained conform to the general theory in Chapter I, we have to prove that they satisfy eqs. (19) and (20).

We have, since $k,\ l \neq 5$:

$$(n\,|\,\bar{\beta}_k\bar{\beta}_l\,|\,n') = \sum_{n''=1}^{5} (n\,|\,\bar{\beta}_k\,|\,n'')\,(n''\,|\,\bar{\beta}_l\,|\,n') =$$

$$= -\sum_{n''=1}^{5} (\delta_{nk}\,\delta_{n''5} - \delta_{n5}\,\delta_{n''k})\,(\delta_{n''l}\,\delta_{n'5} - \delta_{n''5}\,\delta_{n'l}) =$$

$$= \delta_{nk}\,\delta_{n'l} + \delta_{n5}\,\delta_{n'5}\,\delta_{kl}.$$

By interchanging k and l we get:

$$(n\,|\,\bar{\beta}_l\bar{\beta}_k\,|\,n') = \delta_{nl}\,\delta_{n'k} + \delta_{n5}\,\delta_{n'5}\,\delta_{kl}.$$

We have thus:

$$(n\,|\,\beta_k\bar{\beta}_l - \beta_l\bar{\beta}_k\,|\,n') = \delta_{nk}\,\delta_{n'l} - \delta_{nl}\,\delta_{n'k} = (n\,|\,\bar{q}_{kl}\,|\,n'), \tag{83}$$

which is equ. (19) in matrix element form.

To prove equ. (20) we calculate

$$(n\,|\,\bar{\beta}_i\,\bar{\beta}_k\,\bar{\beta}_l\,|\,n') = \sum_{n''=1}^{5}\,\sum_{n'''=1}^{5}(n\,|\,\bar{\beta}_i\,|\,n'')\,(n''\,|\,\bar{\beta}_k\,|\,n''')\,(n'''\,|\,\bar{\beta}_l\,|\,n') =$$

$$= -\,i\sum_{n''=1}^{5}\,\sum_{n'''=1}^{5}(\delta_{ni}\,\delta_{n''5} - \delta_{n5}\,\delta_{n''i})\,(\delta_{n''k}\,\delta_{n'''5} - \delta_{n''5}\,\delta_{n'''k})\,(\delta_{n'''l}\,\delta_{n'5} - \delta_{n'''5}\,\delta_{n'l}) =$$

$$= -\,i\sum_{n'''=1}^{5}(-\,\delta_{ni}\,\delta_{n'''k} - \delta_{n5}\,\delta_{n'''5}\,\delta_{ik})\,(\delta_{n'''l}\,\delta_{n'5} - \delta_{n'''5}\,\delta_{n'l}) =$$

$$= i\,(\delta_{ni}\,\delta_{lk}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'l}\,\delta_{ik}). \tag{84}$$

From this result, we can immediately infer DUFFIN's [9] formula

$$\bar{\beta}_i\,\bar{\beta}_k\,\bar{\beta}_l = 0 \qquad (i \neq k \neq l \neq i). \tag{85}$$

If we interchange i and l in equ. (84) we get:

$$(n\,|\,\bar{\beta}_l\,\bar{\beta}_k\,\bar{\beta}_i\,|\,n') = i\,(\delta_{nl}\,\delta_{ik}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'i}\,\delta_{lk}). \tag{86}$$

By adding eqs. (84) and (86) we obtain

$$(n\,|\,\bar{\beta}_i\,\bar{\beta}_k\,\bar{\beta}_l + \bar{\beta}_l\,\bar{\beta}_k\,\bar{\beta}_i\,|\,n') =$$

$$= \delta_{kl}\,i\,(\delta_{ni}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'i}) + \delta_{ik}\,i\,(\delta_{nl}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'l}) =$$

$$= \delta_{kl}\,(n\,|\,\bar{\beta}_i\,|\,n') + \delta_{ik}\,(n\,|\,\bar{\beta}_l\,|\,n'). \tag{87}$$

This equation can be written:

$$\bar{\beta}_i\,\bar{\beta}_k\,\bar{\beta}_l + \bar{\beta}_l\,\bar{\beta}_k\,\bar{\beta}_i = \bar{\beta}_i\,\delta_{kl} + \bar{\beta}_l\,\delta_{ik}\,. \tag{88}$$

This is PETIAU's well-known commutation rule [28], which DUFFIN and KEMMER took as a starting point for their investigation of the algebra of the $\bar{\beta}_k$:s. Equ. (88) is fulfilled not only by the $\bar{\beta}_k$:s but also by the 10-rowed β_k:s of the vector meson theory, the analogous representation of which we will investigate in the next chapter. Equ. (85), however, holds only for the $\bar{\beta}_k$:s. As has been shown by PETIAU [29], equ. (20) can be proved from equ. (88), e. g. by subtracting equ. (88) with interchanged k and l from the original equation.

We have thus proved that our representation conforms to the general theory and is equivalent to KEMMER's 5-rowed β_k:s. In fact we have

$$\bar{\beta}_k = U^{-1}\,\beta_k\,U \qquad (k = 1, 2, 3, 4), \tag{89}$$

where β_k is the explicit representation given by KEMMER and

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (U^{-1})^*. \tag{90}$$

On the other hand, our representation is closely related to a representation, given by PETIAU [30], in which the matrices contain only real elements.

As has been proved by PETIAU [29] from equ. (88) the quantity

$$\beta = 1 - 2\bar{\beta}_4^2$$

satisfies the conditions for Γ of eqs. (27), (28) and (29).

As is easily seen by means of equ. (80) we have in our representation:

$$(n\,|\,\bar{\beta}\,|\,n') = \delta_{nn'} - 2\,\delta_{n4}\,\delta_{n'4} - 2\,\delta_{n5}\,\delta_{n'5}\,, \tag{91}$$

or

$$\bar{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{92}$$

According to equ. (33) this means that

$$\tilde{\Psi}_k = (\Psi^*\bar{\beta})_k = \Psi_k^* \qquad (k = 1, 2, 3), \tag{93}$$

$$\tilde{\Psi}_k = (\Psi^*\bar{\beta})_k = -\Psi_k^* \qquad (k = 4, 5). \tag{94}$$

For the spatial reflexion of equ. (38) we have by equ. (92) to put

$$Q = -\beta.$$

We will now examine the term λ of equ. (1). As in Chapter II we divide λ into two parts,

$$\bar{\lambda} = \bar{\vartheta} + \bar{\pi}, \tag{95}$$

where $\bar{\vartheta}$ is assumed to contain only coefficients like the β_k:s, which are not altered by a LORENTZ transformation, and where $\bar{\pi}$ is assumed to contain both such coefficients and tensors, and we have thus by equ. (8)

$$Q^{-1}\,\bar{\vartheta}\,Q = \bar{\vartheta} \tag{96}$$

and

$$Q^{-1}\,\bar{\pi}'\,Q = \bar{\pi}. \tag{97}$$

As has been pointed out by HARISH-CHANDRA [12] the most general form of ϑ is a polynomial in $\sum_{k=1}^{4} \beta_k \beta_k$. As a preparation for similar arguments in Chapter V we shall here reproduce and amplify his proof of this statement.

Since, as has been pointed out by KEMMER, any 5-rowed matrix can be expressed as a linear function of the β_k:s and their products, ϑ can be considered as a function of the $\bar{\beta}_k$:s only. As equ. (96) implies that $\bar{\vartheta}$ is an invariant when the β_k:s are considered as vector components, the most general form of ϑ must be:

$$\cdots \sum_{k=1}^{4} \cdots \sum_{l=1}^{4} \cdots \bar{\beta}_k \cdots \bar{\beta}_l \cdots \bar{\beta}_l \cdots \bar{\beta}_k \cdots, \tag{98}$$

where all summations are taken over double indices, but where the indices in general can have any order in the products, so that two β_k:s with the same index are not necessarily neighbours. By using equ. (88) we can move one $\bar{\beta}_k$ two steps to the left say. At the same time we get in general, according to the right hand side of equ. (88), terms which, after performing the summation over one of the indices of the Kronecker symbols, turn out to be of the type (98) but with two β_k:s less than the original expression. When we repeat this procedure with the two $\bar{\beta}_k$:s of a given index, we get, trying to move the two β_k:s as far as possible to the left, one of the following alternatives:

$$\left.\begin{array}{l} \beta_k \beta_k \ldots, \\ \beta_k \bar{\beta}_l \beta_k \ldots, \\ \bar{\beta}_l \bar{\beta}_k \bar{\beta}_m \bar{\beta}_k \ldots \end{array}\right\} \tag{99}$$

Since by equ. (88)

$$\beta_k \bar{\beta}_l \bar{\beta}_k = \bar{\beta}_k \delta_{kl}, \tag{100}$$

the second and third alternatives are reduced to expressions of the type (98) but with two $\bar{\beta}_k$:s less. If we repeat this procedure with the other index pairs and with all the terms with fewer β_k:s, we get finally a sum of terms of the type

$$\sum_{k=1}^{4} \sum_{l=1}^{4} \sum_{m=1}^{4} \cdots \bar{\beta}_k \bar{\beta}_k \bar{\beta}_l \beta_l \bar{\beta}_m \bar{\beta}_m \cdots = \left(\sum_{k=1}^{4} \bar{\beta}_k \bar{\beta}_k \right)^n. \tag{101}$$

The investigation of the $\bar{\vartheta}$ is thus reduced to the study of the expression

$$\sum_{k=1}^{4} \bar{\beta}_k \bar{\beta}_k,$$

which we will calculate in our representation:

$$\left(n\left|\sum_{k=1}^{4}\bar{\beta}_k\beta_k\right|n'\right)=\sum_{k=1}^{4}\sum_{n''=1}^{5}(n|\bar{\beta}_k|n'')\,(n''|\beta_k|n')=$$

$$=-\sum_{k=1}^{4}\sum_{n''=1}^{5}(\delta_{nk}\,\delta_{n''5}-\delta_{n5}\,\delta_{n''k})\,(\delta_{n''k}\,\delta_{n'5}-\delta_{n''5}\,\delta_{n'k})=$$

$$=\sum_{k=1}^{4}(\delta_{nk}\,\delta_{n'k}+\delta_{n5}\,\delta_{n'5})=\sum_{k=1}^{5}\delta_{nk}\,\delta_{n'k}+3\,\delta_{n5}\,\delta_{n'5}\,.$$

Thus:

$$\left(n\left|\sum_{k=1}^{4}\bar{\beta}_k\bar{\beta}_k\right|n'\right)=\delta_{nn'}+3\,\delta_{n5}\,\delta_{n'5}\,, \tag{102}$$

or

$$\sum_{k=1}^{4}\beta_k\,\beta_k=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&4\end{pmatrix}. \tag{103}$$

Let us introduce two matrices $\bar{\omega}_5$ and $\bar{\omega}_4$, the β and γ of HARISH-CHANDRA [11], [13], defined by

$$\bar{\omega}_5=\tfrac{1}{3}\left(\sum_{k=1}^{4}\bar{\beta}_k\bar{\beta}_k-1\right) \tag{104}$$

and

$$\bar{\omega}_4=1-\bar{\omega}_5=-\tfrac{1}{3}\left(\sum_{k=1}^{4}\bar{\beta}_k\bar{\beta}_k-4\right), \tag{105}$$

or

$$\bar{\omega}_5=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix} \tag{106}$$

and

$$\bar{\omega}_4=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix}. \tag{107}$$

$\bar\omega_5$ has evidently the property of only retaining the component with index 5 when operating on the column matrix (78), while $\bar\omega_4$ only retains the components with index up to 4.

It is now easy to see that any power of $\sum_{k=1}^{4}\beta_k\bar\beta_k$, and thus any sum of such powers, can be written as a linear combination of $\bar\omega_4$ and $\bar\omega_5$, and the most general form of $\bar\vartheta$ is consequently:

$$\bar\vartheta = \bar\vartheta_4\,\bar\omega_4 + \bar\vartheta_5\,\bar\omega_5, \tag{108}$$

where $\bar\vartheta_4$ and $\bar\vartheta_5$ are ordinary numbers. At the end of Chapter V we will prove, that if $\bar\vartheta_4 \neq 0$ and $\bar\vartheta_5 \neq 0$, we can without lack of generality assume $\bar\vartheta_4 = \bar\vartheta_5$.

We will now write down equ. (1) in component form:

$$\sum_{n'=1}^{5}\sum_{k=1}^{4} i\,(\delta_{nk}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'k})\,p_k\,\Psi_{n'} + i\,\bar\mu_4\,c\,(\bar\omega_4\,\Psi)_n + i\,\bar\mu_5\,c\,(\bar\omega_5\,\Psi)_n - (\bar\pi\,\Psi)_n = 0, \tag{109}$$

where we have put

$$\bar\vartheta_4 = -\,i\,\bar\mu_4\,c \tag{110}$$

and

$$\bar\vartheta_5 = -\,i\,\bar\mu_5\,c, \tag{111}$$

in conformity with equ. (34).

Equ. (109) gives:

$$p_n\,\Psi_5 = -\,\bar\mu_4\,c\,\Psi_n - i\,(\bar\pi\,\Psi)_n \qquad (n = 1, 2, 3, 4) \tag{112}$$

and with $n = 5$:

$$\sum_{k=1}^{4} p_k\,\Psi_k = \bar\mu_5\,c\,\Psi_5 + i\,(\bar\pi\,\Psi)_5. \tag{113}$$

If we insert

$$\Psi_n = -\,\frac{1}{\bar\mu_4\,c}\,p_n\,\Psi_5 - \frac{i}{\bar\mu_4\,c}\,(\bar\pi\,\Psi)_n \qquad (n = 1, 2, 3, 4) \tag{114}$$

in equ. (113) we get:

$$\left(\sum_{k=1}^{4} p_k^2 + \bar\mu_4\,\bar\mu_5\,c^2\right)\Psi_5 = -\,i\,\bar\mu_4\,c\,(\bar\pi\,\Psi)_5 - i\sum_{k=1}^{4} p_k\,(\bar\pi\,\Psi)_k. \tag{115}$$

Putting

$$\bar\varkappa = \frac{\bar\mu\,c}{\hbar}, \tag{116}$$

where

$$\bar\mu = \sqrt{\bar\mu_4\,\bar\mu_5} \tag{117}$$

and making use of eqs. (2) and (3) we have from eqs. (115) and (114):

$$\left(\sum_{k=1}^{4} \frac{\partial^2}{\partial x_k^2} - \bar{\varkappa}^2\right) \Psi_5 = \frac{i\,\bar{\mu}_4 c}{\hbar^2} (\bar{\pi}\,\Psi)_5 + \frac{1}{\hbar} \sum_{k=1}^{4} \frac{\partial}{\partial x_k} (\bar{\pi}\,\Psi)_k, \tag{118}$$

$$\Psi_n = \frac{i\,\hbar}{\bar{\mu}_4 c} \frac{\partial \Psi_5}{\partial x_n} - \frac{i}{\bar{\mu}_4 c} (\bar{\pi}\,\Psi)_n \quad (n = 1, 2, 3, 4). \tag{119}$$

Equ. (119) for $n = 1, 2, 3$ are to be regarded as equations of definition for the redundant components Ψ_1, Ψ_2, Ψ_3. These components are proportional to the components of the gradient of Ψ_5 when

$$\bar{\pi} = 0, \tag{120}$$

in which case equ. (118) is identical with KLEIN-GORDON's equation.

Let us now consider the adjoint equ. (22):

$$\sum_{n'=1}^{5} \sum_{k=1}^{4} i\,\tilde{\Psi}_{n'} (\delta_{n'k}\,\delta_{n5} - \delta_{n'5}\,\delta_{nk})\, p_k + i\,\bar{\mu}_4 c\,(\tilde{\Psi}\,\bar{\omega}_4)_n + i\,\bar{\mu}_5 c\,(\tilde{\Psi}\,\bar{\omega}_5)_n - (\tilde{\Psi}\,\bar{\pi})_n = 0, \tag{121}$$

which gives:

$$\tilde{\Psi}_5\, p_n = \bar{\mu}_4 c\,\tilde{\Psi}_n + i\,(\tilde{\Psi}\,\bar{\pi})_n \quad (n = 1, 2, 3, 4) \tag{122}$$

and for $n = 5$:

$$\sum_{k=1}^{4} \tilde{\Psi}_k\, p_k = -\,\bar{\mu}_5 c\,\tilde{\Psi}_5 - i\,(\tilde{\Psi}\,\bar{\pi})_5. \tag{123}$$

If we insert

$$\tilde{\Psi}_n = \frac{1}{\bar{\mu}_4 c}\,\tilde{\Psi}_5\, p_n - \frac{i}{\bar{\mu}_4 c} (\tilde{\Psi}\,\bar{\pi})_n \quad (n = 1, 2, 3, 4) \tag{124}$$

in equ. (123) we obtain:

$$\tilde{\Psi}_5 \left(\sum_{k=1}^{4} p_k^2 + \bar{\mu}_4\,\bar{\mu}_5\, c^2\right) = -\,i\,\bar{\mu}_4 c\,(\tilde{\Psi}\,\bar{\pi})_5 + i \sum_{k=1}^{4} (\tilde{\Psi}\,\bar{\pi})_k\, p_k. \tag{125}$$

By eqs. (116), (117), (23) and (24) we obtain:

$$\left(\sum_{k=1}^{4} \frac{\partial^2}{\partial x_k^2} - \bar{\varkappa}^2\right) \tilde{\Psi}_5 = \frac{i\,\bar{\mu}_4 c}{\hbar^2} (\tilde{\Psi}\,\bar{\pi})_5 + \frac{1}{\hbar} \sum_{k=1}^{4} \frac{\partial}{\partial x_k} (\tilde{\Psi}\,\bar{\pi})_k \tag{126}$$

and

$$\tilde{\Psi}_n = \frac{i\,\hbar}{\bar{\mu}_4 c} \frac{\partial \tilde{\Psi}_5}{\partial x_n} - \frac{i}{\bar{\mu}_4 c} (\tilde{\Psi}\,\bar{\pi})_n \quad (n = 1, 2, 3, 4). \tag{127}$$

With the help of eqs. (33), (92) and (34) it is easy to show, that eqs. (126) and (127) are actually the conjugate complex of eqs. (118) and (119).

Finally we wish to discuss the possibility of assuming Ψ as a 5-dimensional antisymmetrical tensor of the 4th rank, i. e. a 5-dimensional pseudovector Ψ_k. If we restrict ourselves to transformations in the 4-dimensional subspace ($\varepsilon_{k5} = 0$), this means that Ψ_1, Ψ_2, Ψ_3, Ψ_4 transform like a 4-dimensional pseudovector and Ψ_5 like a pseudoscalar. The transformation formula of a pseudovector Ψ_k is for infinitesimal transformations exactly the same as if it were an ordinary vector, i. e. equ. (72) and the subsequent deduction of the matrix representation are still valid. This means, as is well-kown, that the 5-rowed $\bar{\beta}_k$:s are, even in this case, the proper representation of the infinitesimal transformations, if inserted in eqs. (21) and (6).

For the spatial reflexion of equ. (38) we have in this case to put

$$Q = \bar{\beta}. \tag{128}$$

CHAPTER IV

The vector and pseudovector meson theory

In this chapter we will investigate the vector meson case along the same lines as in the treatment of the scalar meson theory in the preceding chapter, thus deriving a similar representation of the 10-rowed matrices of DUFFIN and KEMMER.

We assume that Ψ is transformed like a 5-dimensional, antisymmetrical tensor Ψ_{kl} of the second rank. We then have:

$$\Psi'_{kl} = \Psi_{kl} + \sum_{s=1}^{5} (\varepsilon_{ks} \Psi_{sl} + \varepsilon_{ls} \Psi_{ks}). \tag{129}$$

On the other hand, denoting all matrices belonging to this case by a double bar, equ. (6) gives:

$$\Psi'_{kl} = \left[\left(1 + \tfrac{1}{2} \sum_{r=1}^{5} \sum_{s=1}^{5} \varepsilon_{rs} \bar{\bar{q}}_{rs}\right) \Psi\right]_{kl}, \tag{130}$$

which equation is supposed to be identical with equ. (129). We can write equ. (129) in the following manner:

$$\Psi'_{kl} = \Psi_{kl} + \tfrac{1}{2} \sum_{r=1}^{5} \sum_{s=1}^{5} \varepsilon_{rs} (\delta_{kr} \Psi_{sl} - \delta_{ks} \Psi_{rl} + \delta_{lr} \Psi_{ks} - \delta_{ls} \Psi_{kr}), \tag{131}$$

and equ. (130) implies

$$\Psi'_{kl} = \Psi_{kl} + \tfrac{1}{2} \sum_{r=1}^{5} \sum_{s=1}^{5} \varepsilon_{rs} (\bar{\bar{q}}_{rs} \Psi)_{kl}. \tag{132}$$

Identification of eqs. (131) and (132) gives:

$$(q_{rs} \Psi)_{kl} = \delta_{kr} \Psi_{sl} - \delta_{ks} \Psi_{rl} + \delta_{rl} \Psi_{ks} - \delta_{ls} \Psi_{kr}, \tag{133}$$

which equation, like equ. (76), has been derived by KLEIN in the unpublished manuscript already quoted.

Now we proceed to write equ. (133) in the following manner:

$$(\bar{\bar{q}}_{rs}\,\Psi)_{kl} = \tfrac{1}{2}\sum_{m=1}^{5}\sum_{n=1}^{5}(\delta_{kr}\,\delta_{ms}\,\delta_{ln} - \delta_{ks}\,\delta_{mr}\,\delta_{ln} - \delta_{lr}\,\delta_{ms}\,\delta_{kn} + \delta_{ls}\,\delta_{mr}\,\delta_{kn} - \delta_{kr}\,\delta_{ns}\,\delta_{lm} + \delta_{ks}\,\delta_{nr}\,\delta_{lm} + \delta_{lr}\,\delta_{ns}\,\delta_{km} - \delta_{ls}\,\delta_{nr}\,\delta_{km})\,\Psi_{mn}. \tag{134}$$

Let us consider Ψ as a column matrix,

$$\Psi = \begin{vmatrix} \Psi_{23} \\ \Psi_{31} \\ \Psi_{12} \\ \Psi_{41} \\ \Psi_{42} \\ \Psi_{43} \\ \Psi_{51} \\ \Psi_{52} \\ \Psi_{53} \\ \Psi_{54} \end{vmatrix}, \tag{135}$$

containing the 10 independent elements of the antisymmetrical tensor Ψ_{kl}. Since in the double sum of equ. (134) both the bracket and Ψ_{mn} are antisymmetrical in the index pair mn, the index pairs occur twice each and with the same sign. The double sum of equ. (134) is thus 2 times the sum only over the index pairs of equ. (135), which sum we will denote by $\sum\limits_{mn}$. We can thus write equ. (134):

$$(\bar{\bar{q}}_{kl}\,\Psi)_{mn} = \sum_{m'n'}(mn\,|\,\bar{\bar{q}}_{kl}\,|\,m'n')\,\Psi_{m'n'}, \tag{136}$$

where

$$(mn\,|\,\bar{\bar{q}}_{kl}\,|\,m'n') = \delta_{mk}\,\delta_{m'l}\,\delta_{nn'} - \delta_{ml}\,\delta_{m'k}\,\delta_{nn'} - \delta_{nk}\,\delta_{m'l}\,\delta_{mn'} + \delta_{nl}\,\delta_{m'k}\,\delta_{mn'} - \delta_{mk}\,\delta_{n'l}\,\delta_{nm'} + \delta_{ml}\,\delta_{n'k}\,\delta_{nm'} + \delta_{nk}\,\delta_{n'l}\,\delta_{mm'} - \delta_{nl}\,\delta_{n'k}\,\delta_{mm'} \tag{137}$$

can be interpreted as the matrix element of q_{kl}. We can write this expression for short:

$$(mn\,|\,\bar{\bar{q}}_{kl}\,|\,m'n') = \delta_{mk}\,\delta_{m'l}\,\delta_{nn'} - \mathrm{ex}\,(kl) - \mathrm{ex}\,(mn) - \mathrm{ex}\,(m'n'), \tag{138}$$

where ex (kl) means the entire preceding expression with k and l exchanged.

If β_k are the 10-rowed Γ_k:s in this case, equ. (18) gives:

$$(mn\,|\,\bar{\bar{\beta}}_k\,|\,m'n') = i\,(\delta_{mk}\,\delta_{m'5}\,\delta_{nn'} - \mathrm{ex}\,(k5) - \mathrm{ex}\,(mn) - \mathrm{ex}\,(m'n')). \tag{139}$$

Of course we can begin with any other of the 8 terms in equ. (139) e. g.

$$(mn\,|\,\bar{\bar{\beta}}_k\,|\,m'n') = i\,(-\,\delta_{m'k}\,\delta_{m5}\,\delta_{nn'} - \mathrm{ex}\,(k5) - \mathrm{ex}\,(mn) - \mathrm{ex}\,(m'n')), \tag{140}$$

from which we obtain:

$$\begin{aligned}(mn\,|\,\bar{\beta}_k\,|\,m'n')^* &= i\,(\delta_{m'k}\,\delta_{m5}\,\delta_{nn'} - \mathrm{ex}\,(k5) - \mathrm{ex}\,(mn) - \mathrm{ex}\,(m'n')) = \\ &= (m'n'\,|\,\bar{\bar{\beta}}_k\,|\,mn),\end{aligned} \tag{141}$$

i. e. the $\bar{\beta}_k$:s are Hermitian.

The form of the matrices are, using the indices of equ. (135)

$$\bar{\bar{\beta}}_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-i&0\\ 0&0&0&0&0&0&0&i&0&0\\ 0&0&0&0&0&0&0&0&0&-i\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&-i&0&0&0&0&0&0&0\\ 0&i&0&0&0&0&0&0&0&0\\ 0&0&0&i&0&0&0&0&0&0 \end{pmatrix}, \quad \bar{\bar{\beta}}_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&i&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-i&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&-i\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&i&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ -i&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&i&0&0&0&0&0 \end{pmatrix},$$

$$\beta_3 = \begin{pmatrix} 0&0&0&0&0&0&0&-i&0&0\\ 0&0&0&0&0&0&i&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&-i\\ 0&-i&0&0&0&0&0&0&0&0\\ i&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&i&0&0&0&0 \end{pmatrix}, \quad \bar{\beta}_4 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&i&0&0&0\\ 0&0&0&0&0&0&0&i&0&0\\ 0&0&0&0&0&0&0&0&i&0\\ 0&0&0&-i&0&0&0&0&0&0\\ 0&0&0&0&-i&0&0&0&0&0\\ 0&0&0&0&0&-i&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}. \tag{142}$$

In this case too, the Hermitian matrices have only purely imaginary elements, so that the matrices alter their sign when they are transposed.

In order to prove that the matrices conform to the general theory, we will prove equ. (19) and PETIAU's commutation rule, from which follows equ. (20).

At first sight the eight terms in the matrix elements might seem a bit cumbersome to handle in calculating the matrix products. As a matter of fact we can by a simple artifice reduce the number of terms in the matrix elements to four or less. The method will be clear from the following two examples, which we perform in considerable detail:

$$(mn\,|\,\bar{\bar{\beta}}_k\bar{\bar{\beta}}_l-\bar{\bar{\beta}}_l\bar{\bar{\beta}}_k\,|\,m'n')=\sum_{m''n''}(mn\,|\,\beta_k\,|\,m''n'')\,(m''n''\,|\,\beta_l\,|\,m'n')-\mathrm{ex}\,(kl)=$$

$$=-\sum_{m''n''}(\delta_{mk}\,\delta_{m''5}\,\delta_{nn''}-\mathrm{ex}\,(k5)-\mathrm{ex}\,(mn)-\mathrm{ex}\,(m''n''))\cdot$$

$$\cdot\,(\delta_{m''l}\,\delta_{m'5}\,\delta_{n''n'}-\mathrm{ex}\,(l5)-\mathrm{ex}\,(m''n'')-\mathrm{ex}\,(m'n'))-\mathrm{ex}\,(kl)=$$

$$=-\tfrac{1}{2}\sum_{m''=1}^{5}\sum_{n''=1}^{5}(\delta_{mk}\,\delta_{m''5}\,\delta_{nn''}-\mathrm{ex}\,(k5)-\mathrm{ex}\,(m''n''))\cdot$$

$$\cdot\,(\delta_{m''l}\,\delta_{m'5}\,\delta_{n''n'}-\mathrm{ex}\,(l5)-\mathrm{ex}\,(m''n''))-\mathrm{ex}\,(mn)-\mathrm{ex}\,(m'n')-\mathrm{ex}\,(kl)=$$

$$=-\sum_{m''=1}^{5}\sum_{n''=1}^{5}(\delta_{mk}\,\delta_{m''5}\,\delta_{nn''}-\mathrm{ex}\,(k5))\,(\delta_{m''l}\,\delta_{m'5}\,\delta_{n''n'}-\mathrm{ex}\,(l5)-\mathrm{ex}\,(m''n''))-$$

$$-\,\mathrm{ex}\,(mn)-\mathrm{ex}\,(m'n')-\mathrm{ex}\,(kl)=$$

$$=-\sum_{m''=1}^{5}\sum_{n''=1}^{5}(\delta_{mk}\,\delta_{m''5}\,\delta_{nn''}-\delta_{m5}\,\delta_{m''k}\,\delta_{nn''})\cdot$$

$$\cdot\,(\delta_{m''l}\,\delta_{m'5}\,\delta_{n''n'}-\delta_{m''5}\,\delta_{m'l}\,\delta_{n''n'}-\delta_{n''l}\,\delta_{m'5}\,\delta_{m''n'}+\delta_{n''5}\,\delta_{m'l}\,\delta_{m''n'})-$$

$$-\,\mathrm{ex}\,(mn)-\mathrm{ex}\,(m'n')-\mathrm{ex}\,(kl)=$$

$$=\delta_{nn'}\,\delta_{mk}\,\delta_{m'l}+\delta_{5n'}\,\delta_{nl}\,\delta_{mk}\,\delta_{m'5}-\delta_{5n'}\,\delta_{5n}\,\delta_{mk}\,\delta_{m'l}+\delta_{lk}\,\delta_{nn'}\,\delta_{m5}\,\delta_{m'5}-$$

$$-\,\delta_{\lambda n'}\,\delta_{ln}\,\delta_{m5}\,\delta_{m'5}+\delta_{\lambda n'}\,\delta_{n5}\,\delta_{m5}\,\delta_{m'l}-\mathrm{ex}\,(mn)-\mathrm{ex}\,(m'n')-\mathrm{ex}\,(kl).\qquad(143)$$

Here the 2nd, 4th and 6th terms vanish when $\mathrm{ex}\,(m'n')$, $\mathrm{ex}\,(kl)$ and $\mathrm{ex}\,(mn)$ resp. are subtracted. Since instead of any term we can always write any term resulting from the exchanges of the original term, we can instead of the 3rd term write $+\delta_{5m'}\,\delta_{5m}\,\delta_{nl}\,\delta_{n'k}$, which cancels the 5th term. The result is:

$$(mn\,|\,\bar{\bar{\beta}}_k\bar{\bar{\beta}}_l-\bar{\bar{\beta}}_l\bar{\bar{\beta}}_k\,|\,m'n')=$$

$$=\delta_{mk}\,\delta_{m'l}\,\delta_{nn'}-\mathrm{ex}\,(kl)-\mathrm{ex}\,(mn)-\mathrm{ex}\,(m'n')=(mn\,|\,\bar{\bar{q}}_{kl}\,|\,m'n'),\qquad(144)$$

which is equ. (19).

We will now proceed to prove PETIAU's commutation rule:

$$(mn\,|\,\bar{\bar{\beta}}_i\bar{\bar{\beta}}_k\bar{\bar{\beta}}_l+\bar{\bar{\beta}}_l\bar{\bar{\beta}}_k\bar{\beta}_i\,|\,m'n')=$$

$$=-i\sum_{m''=1}^{5}\sum_{n''=1}^{5}\sum_{m'''=1}^{5}\sum_{n'''=1}^{5}(\delta_{mi}\,\delta_{m''5}\,\delta_{nn''}-\delta_{m5}\,\delta_{m''i}\,\delta_{nn''}-\delta_{mi}\,\delta_{n''5}\,\delta_{nm''}+\delta_{m5}\,\delta_{n''i}\,\delta_{nm''})\cdot$$

$$\cdot\, (\delta_{m''k}\,\delta_{m'''5}\,\delta_{n''n'''} - \delta_{m''5}\,\delta_{m'''k}\,\delta_{n''n'''})\,\cdot$$
$$\cdot\, (\delta_{m'''l}\,\delta_{m'5}\,\delta_{n'''n'} - \delta_{m'''5}\,\delta_{m'l}\,\delta_{n'''n'} - \delta_{n'''l}\,\delta_{m'5}\,\delta_{m'''n'} + \delta_{n'''5}\,\delta_{m'l}\,\delta_{m'''n'}) -$$
$$- \operatorname{ex}(mn) - \operatorname{ex}(m'n') + \operatorname{ex}(il) =$$
$$= -i \sum_{m'''-1}^{5} \sum_{n'''-1}^{5} (- \delta_{nn'''}\,\delta_{mi}\,\delta_{m'''k} - \delta_{ik}\,\delta_{nn'''}\,\delta_{m5}\,\delta_{m'''5} - \delta_{nk}\,\delta_{5n'''}\,\delta_{mi}\,\delta_{m'''5} +$$
$$+ \delta_{n5}\,\delta_{5n'''}\,\delta_{mi}\,\delta_{m'''k} + \delta_{nk}\,\delta_{in'''}\,\delta_{m5}\,\delta_{m'''5} - \delta_{n5}\,\delta_{in'''}\,\delta_{m5}\,\delta_{m'''k})\,\cdot$$
$$\cdot\, (\delta_{m'''l}\,\delta_{m'5}\,\delta_{n'''n'} - \delta_{m'''5}\,\delta_{m'l}\,\delta_{n'''n'} - \delta_{n'''l}\,\delta_{m'5}\,\delta_{m'''n'} + \delta_{n'''5}\,\delta_{m'l}\,\delta_{m'''n'}) -$$
$$- \operatorname{ex}(mn) - \operatorname{ex}(m'n') + \operatorname{ex}(il) =$$
$$= -i\,(- \delta_{kl}\,\delta_{nn'}\,\delta_{mi}\,\delta_{m'5} + \delta_{kn'}\,\delta_{nl}\,\delta_{mi}\,\delta_{m'5} - \delta_{kn'}\,\delta_{n5}\,\delta_{mi}\,\delta_{m'l} +$$
$$+ \delta_{nn'}\,\delta_{ik}\,\delta_{m5}\,\delta_{m'l} + \delta_{5n'}\,\delta_{nl}\,\delta_{ik}\,\delta_{m5}\,\delta_{m'5} - \delta_{5n'}\,\delta_{n5}\,\delta_{ik}\,\delta_{m5}\,\delta_{m'l} +$$
$$+ \delta_{5n'}\,\delta_{nk}\,\delta_{mi}\,\delta_{m'l} - \delta_{5n'}\,\delta_{nk}\,\delta_{mi}\,\delta_{m'l} + \delta_{kl}\,\delta_{5n'}\,\delta_{n5}\,\delta_{mi}\,\delta_{m'5} +$$
$$+ \delta_{kn'}\,\delta_{n5}\,\delta_{mi}\,\delta_{m'l} - \delta_{in'}\,\delta_{nk}\,\delta_{m5}\,\delta_{m'l} - \delta_{5n'}\,\delta_{il}\,\delta_{nk}\,\delta_{m5}\,\delta_{m'5} -$$
$$- \delta_{kl}\,\delta_{in'}\,\delta_{n5}\,\delta_{m5}\,\delta_{m'5} + \delta_{kn'}\,\delta_{il}\,\delta_{n5}\,\delta_{m5}\,\delta_{m'5}) +$$
$$+ \operatorname{ex}(il) - \operatorname{ex}(mn) - \operatorname{ex}(m'n') =$$
$$= \delta_{kl}\,i\,\delta_{mi}\,\delta_{m'5}\,\delta_{nn'} - \delta_{ik}\,i\,\delta_{m5}\,\delta_{m'l}\,\delta_{nn'} + \operatorname{ex}(il) - \operatorname{ex}(mn) - \operatorname{ex}(m'n') =$$
$$= \delta_{kl}\,i\,(\delta_{mi}\,\delta_{m'5}\,\delta_{nn'} - \delta_{m5}\,\delta_{m'i}\,\delta_{nn'}) - \operatorname{ex}(mn) - \operatorname{ex}(m'n') +$$
$$+ \delta_{ik}\,i\,(\delta_{ml}\,\delta_{m'5}\,\delta_{nn'} - \delta_{m5}\,\delta_{m'l}\,\delta_{nn'}) - \operatorname{ex}(mn) - \operatorname{ex}(m'n') =$$
$$= \delta_{kl}\,(mn\,|\,\bar{\bar{\beta}}_i\,|\,m'n') + \delta_{ik}\,(mn\,|\,\bar{\bar{\beta}}_l\,|\,m'n'), \tag{145}$$

or

$$\bar{\bar{\beta}}_i\,\bar{\bar{\beta}}_k\,\bar{\beta}_l + \bar{\bar{\beta}}_l\,\bar{\bar{\beta}}_k\,\bar{\bar{\beta}}_i = \bar{\bar{\beta}}_i\,\delta_{kl} + \bar{\bar{\beta}}_l\,\delta_{ik}\,, \tag{146}$$

which is PETIAU's commutation rule, and the conformity with the general scheme is thus established.

Our β_k:s must hence be equivalent to KEMMERS 10-rowed β_k:s, and as a matter of fact we have:

$$\beta_k = U^{-1}\,\beta_k\,U \qquad (k = 1, 2, 3, 4), \tag{147}$$

where β_k is the explicit 10-rowed representation given by KEMMER and

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix} = (U^{-1})^*. \tag{148}$$

Other similar representations have been given by PETIAU [30] and SCHRÖDINGER [31], which can easily be proved to be equivalent with ours.

In this case too the quantity

$$\bar\beta = 1 - 2\,\bar{\bar\beta}_4^2 \tag{149}$$

satisfies the conditions for Γ of eqs. (27), (28) and (29). Since the unit matrix can be written as $\frac{1}{2}\,\delta_{mm'}\,\delta_{nn'} - \mathrm{ex}\,(mn) - \mathrm{ex}\,(m'n')$, we easily find by equ. (139):

$$(mn\,|\,\bar\beta\,|\,m'n') = \tfrac{1}{2}\,\delta_{mm'}\,\delta_{nn'} - 2\,\delta_{nn'}\,\delta_{m4}\,\delta_{m'4} - 2\,\delta_{nn'}\,\delta_{m5}\,\delta_{m'5} + \\ + 4\,\delta_{m5}\,\delta_{m'5}\,\delta_{n4}\,\delta_{n'4} - \mathrm{ex}\,(mn) - \mathrm{ex}\,(m'n'), \tag{150}$$

or with the indices of equ. (135):

$$\bar{\bar\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{151}$$

From equ. (33) this means that:

$$\tilde\Psi_{kl} = (\Psi^*\,\beta)_{kl} = \Psi^*_{kl} \qquad (k, l = 1, 2, 3), \tag{152}$$

$$\tilde\Psi_{4k} = (\Psi^*\,\beta)_{4k} = -\,\Psi^*_{4k} \qquad (k = 1, 2, 3), \tag{153}$$

$$\tilde\Psi_{5k} = (\Psi^*\,\bar\beta)_{5k} = -\,\Psi^*_{5k} \qquad (k = 1, 2, 3), \tag{154}$$

$$\tilde\Psi_{54} = (\Psi^*\,\bar\beta)_{54} = \Psi^*_{54}\,. \tag{155}$$

For the spatial reflexion of equ. (38) we have by equ. (151) to put

$$Q = \bar\beta.$$

As before we separate $\bar\lambda$ into two parts:

$$\bar{\bar\lambda} = \bar{\bar\vartheta} + \bar\pi\,, \tag{156}$$

where

$$Q^{-1}\,\bar{\bar\vartheta}\,Q = \bar{\bar\vartheta} \tag{157}$$

and

$$Q^{-1}\,\bar{\bar\pi}'\,Q = \bar\pi\,. \tag{158}$$

HARISH-CHANDRA'S proof, reproduced in the preceding chapter, that ϑ could be reduced to a sum of powers of $\sum_{k=1}^{4} \beta_k \bar{\beta}_k$ can be performed in exactly the same manner for the $\bar{\bar{\beta}}_k$:s, since only PETIAU'S commutation rule was used, and this is valid in both cases.

We will thus calculate $\sum_{k=1}^{4} \bar{\bar{\beta}}_k \bar{\beta}_k$ in our representation:

$$
\begin{aligned}
&\left(mn \left| \sum_{k=1}^{4} \beta_k \beta_k \right| m'n' \right) = \\
&= - \sum_{k=1}^{4} \sum_{m''=1}^{5} \sum_{n''=1}^{5} (\delta_{mk}\, \delta_{m''5}\, \delta_{nn''} - \delta_{m5}\, \delta_{m''k}\, \delta_{nn''}) \cdot \\
&\cdot (\delta_{m''k}\, \delta_{m'5}\, \delta_{n'n''} - \delta_{m''5}\, \delta_{m'k}\, \delta_{n'n''} - \delta_{n''k}\, \delta_{m'5}\, \delta_{n'm''} + \delta_{n''5}\, \delta_{m'k}\, \delta_{n'm''}) - \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \operatorname{ex}(mn) - \operatorname{ex}(m'n') = \\
&= - \sum_{k=1}^{4} (-\delta_{nn'}\, \delta_{mk}\, \delta_{m'k} + 2\, \delta_{n'5}\, \delta_{n5}\, \delta_{mk}\, \delta_{m'k} - \delta_{nn'}\, \delta_{m5}\, \delta_{m'5}) - \operatorname{ex}(mn) - \operatorname{ex}(m'n') = \\
&\quad = \sum_{k=1}^{5} (\delta_{nn'}\, \delta_{mk}\, \delta_{m'k} - 2\, \delta_{n'5}\, \delta_{n5}\, \delta_{mk}\, \delta_{m'k}) + \\
&+ 3\, \delta_{nn'}\, \delta_{m5}\, \delta_{m'5} + 2\, \delta_{n'5}\, \delta_{n5}\, \delta_{m5}\, \delta_{m'5} - \operatorname{ex}(mn) - \operatorname{ex}(m'n') = \\
&= \delta_{nn'}\, \delta_{mm'} - 2\, \delta_{mm'}\, \delta_{n'5}\, \delta_{n5} + \\
&\qquad\qquad + 3\, \delta_{nn'}\, \delta_{m5}\, \delta_{m'5} + 2\, \delta_{n'5}\, \delta_{n5}\, \delta_{m5}\, \delta_{m'5} - \operatorname{ex}(mn) - \operatorname{ex}(m'n').
\end{aligned}
$$

We finally obtain:

$$
\left(mn \left| \sum_{k=1}^{4} \bar{\bar{\beta}}_k \bar{\bar{\beta}}_k \right| m'n' \right) = \delta_{nn'}\, \delta_{mm'} + \delta_{nn'}\, \delta_{m5}\, \delta_{m'5} - \operatorname{ex}(mn) - \operatorname{ex}(m'n') \tag{159}
$$

or with the indices of equ. (135):

$$
\sum_{k=1}^{4} \bar{\bar{\beta}}_k \bar{\bar{\beta}}_k = \left\{ \begin{array}{cccccccccc}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{array} \right\} . \tag{160}
$$

Let us again introduce two matrices ω_5 and ω_4, the β and γ of HARISH-CHANDRA, now defined by

$$\bar{\bar{\omega}}_5 = \sum_{k=1}^{4} \bar{\beta}_k \bar{\bar{\beta}}_k - 2 \tag{161}$$

and

$$\bar{\bar{\omega}}_4 = 1 - \bar{\bar{\omega}}_5 = 3 - \sum_{k=1}^{4} \bar{\beta}_k \bar{\bar{\beta}}_k , \tag{162}$$

or

$$\bar{\bar{\omega}}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{163}$$

and

$$\bar{\bar{\omega}}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{164}$$

where ω_5 clearly has the property of only retaining the components of Ψ having an index 5, whereas ω_4 only retains the components of Ψ in which both indices are ≤ 4.

We have consequently, that every $\bar{\bar{\vartheta}}$ can be written in the form:

$$\bar{\bar{\vartheta}} = \bar{\bar{\vartheta}}_4 \bar{\bar{\omega}}_4 + \bar{\bar{\vartheta}}_5 \bar{\bar{\omega}}_5 , \tag{165}$$

where $\bar{\bar{\vartheta}}_4$ and $\bar{\bar{\vartheta}}_5$ are ordinary numbers. At the end of Chapter V we will prove, that if $\bar{\bar{\vartheta}}_4 \neq 0$ and $\bar{\bar{\vartheta}}_5 \neq 0$, we can without lack of generality assume $\vartheta_4 = \bar{\bar{\vartheta}}_5$.

We will now write down equ. (1) in component form:

$$\sum_{m'=1}^{5} \sum_{n'=1}^{5} \sum_{k=1}^{4} i\,(\delta_{mk}\,\delta_{m'5}\,\delta_{nn'} - \delta_{m5}\,\delta_{m'k}\,\delta_{nn'} - \delta_{nk}\,\delta_{m'5}\,\delta_{mn'} + \delta_{n5}\,\delta_{m'k}\,\delta_{mn'})\,p_k\,\Psi_{m'n'} + \\ + i\,\bar{\bar{\mu}}_4\,c\,(\bar{\bar{\omega}}_4\,\Psi)_{mn} + i\,\bar{\bar{\mu}}_5\,c\,(\bar{\bar{\omega}}_5\,\Psi)_{mn} - (\bar{\bar{\pi}}\,\Psi)_{mn} = 0, \qquad (166)$$

where we have put:

$$\vartheta_4 = -\,i\,\bar{\bar{\mu}}_4\,c \qquad (167)$$

and

$$\bar{\bar{\vartheta}}_5 = -\,i\,\bar{\bar{\mu}}_5\,c. \qquad (168)$$

Equ. (166) gives:

$$p_m\,\Psi_{5n} - p_n\,\Psi_{5m} = -\,\bar{\bar{\mu}}_4\,c\,\Psi_{mn} - i\,(\bar{\bar{\pi}}\,\Psi)_{mn} \qquad (m, n = 1, 2, 3, 4) \quad (169)$$

and for $m = 5$:

$$\sum_{k=1}^{4} p_k\,\Psi_{kn} = \bar{\bar{\mu}}_5\,c\,\Psi_{5n} + i\,(\bar{\bar{\pi}}\,\Psi)_{5n} \qquad (n = 1, 2, 3, 4). \quad (170)$$

After multiplying equ. (170) by p_n from the left and summing over n from 1 to 4, the left hand side vanishes, and we get

$$\sum_{n=1}^{4} p_n\,\Psi_{5n} = -\,\frac{i}{\bar{\bar{\mu}}_5\,c} \sum_{n=1}^{4} p_n\,(\bar{\bar{\pi}}\,\Psi)_{5n}. \qquad (171)$$

If we insert

$$\Psi_{kn} = -\,\frac{1}{\bar{\bar{\mu}}_4\,c}\,(p_k\,\Psi_{5n} - p_n\,\Psi_{5k}) - \frac{i}{\bar{\bar{\mu}}_4\,c}\,(\bar{\bar{\pi}}\,\Psi)_{kn} \qquad (k, n = 1, 2, 3, 4) \quad (172)$$

in equ. (170) we obtain by equ. (171):

$$\left(\sum_{k=1}^{4} p_k^2 + \bar{\bar{\mu}}_4\,\bar{\bar{\mu}}_5\,c^2\right)\Psi_{5n} = -\,i\,\bar{\bar{\mu}}_4\,c\,(\bar{\bar{\pi}}\,\Psi)_{5n} - i \sum_{k=1}^{4} p_k\,(\bar{\bar{\pi}}\,\Psi)_{kn} - \\ -\,\frac{i}{\bar{\bar{\mu}}_5\,c} \sum_{k=1}^{4} p_n\,p_k\,(\bar{\bar{\pi}}\,\Psi)_{5k} \qquad (n = 1, 2, 3, 4). \quad (173)$$

For $\pi = 0$ we see that besides the ordinary vector meson equations in the free space, we can, following HARISH-CHANDRA [11], get MAXWELL's equations by putting

$$\bar{\bar{\mu}}_5 = 0, \qquad (174)$$

$$\Psi_{5k} = A_k \qquad (k = 1, 2, 3, 4), \qquad (175)$$

$$\Psi_{mn} = \frac{i\,\hbar}{\bar{\bar{\mu}}_4\,c}\,F_{mn} \qquad (m, n = 1, 2, 3, 4), \qquad (176)$$

where A_k is the potential 4-vector and F_{mn} is the field strength tensor.

If we now put

$$\varkappa = \frac{\bar{\bar{\mu}}\, c}{\hbar}, \tag{177}$$

where

$$\bar{\bar{\mu}} = \sqrt{\bar{\bar{\mu}}_4\, \bar{\bar{\mu}}_5}, \tag{178}$$

and make use of eqs. (2) and (3), we get from equ. (173):

$$\left(\sum_{k=1}^{4} \frac{\partial^2}{\partial x_k^2} - \varkappa^2\right) \Psi_{5n} = \frac{i\, \bar{\bar{\mu}}_4\, c}{\hbar^2} (\pi\, \Psi)_{5n} + \\ + \frac{1}{\hbar} \sum_{k=1}^{4} \frac{\partial}{\partial x_k} (\bar{\bar{\pi}}\, \Psi)_{kn} - \frac{i}{\bar{\bar{\mu}}_5\, c} \frac{\partial}{\partial x_n} \sum_{k=1}^{4} \frac{\partial}{\partial x_k} (\bar{\bar{\pi}}\, \Psi)_{5k} \quad (n = 1, 2, 3, 4), \tag{179}$$

from equ. (172):

$$\Psi_{kn} = \frac{i\hbar}{\bar{\bar{\mu}}_4\, c} \left(\frac{\partial}{\partial x_k} \Psi_{5n} - \frac{\partial}{\partial x_n} \Psi_{5k}\right) - \frac{i}{\bar{\bar{\mu}}_4\, c} (\bar{\bar{\pi}}\, \Psi)_{kn} \qquad (k, n = 1, 2, 3, 4) \tag{180}$$

and from equ. (170) with $n = 4$:

$$\Psi_{54} = \frac{i\hbar}{\bar{\bar{\mu}}_5\, c} \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \Psi_{4k} - \frac{i}{\bar{\bar{\mu}}_5\, c} (\bar{\bar{\pi}}\, \Psi)_{54}. \tag{181}$$

Equ. (179) is obviously equal to PROCA's equation when $\bar{\bar{\pi}} = 0$. Eqs. (180) for $k, n = 1, 2, 3$ and (181) are to be considered as equations of definition for the redundant components Ψ_{kn} $(k, n = 1, 2, 3)$ and Ψ_{54} which, when $\pi = 0$, are proportional to the rotation of Ψ_{5k} $(k = 1, 2, 3)$ and the divergence of Ψ_{4k} respectively.

Let us now consider the adjoint equ. (22):

$$\sum_{m'=1}^{5} \sum_{n'=1}^{5} \sum_{k=1}^{4} i\, \tilde{\Psi}_{m'n'}\, p_k \left(\delta_{m'k}\, \delta_{m5}\, \delta_{nn'} - \delta_{m'5}\, \delta_{mk}\, \delta_{nn'} - \delta_{m'k}\, \delta_{n5}\, \delta_{mn'} + \delta_{m'5}\, \delta_{nk}\, \delta_{mn'}\right) + \\ + i\, \bar{\bar{\mu}}_4\, c\, (\tilde{\Psi}\, \bar{\bar{\omega}}_4)_{mn} + i\, \bar{\bar{\mu}}_5\, c\, (\tilde{\Psi}\, \bar{\bar{\omega}}_5)_{mn} - (\tilde{\Psi}\, \bar{\bar{\pi}})_{mn} = 0, \tag{182}$$

from which we derive the following equations for the components:

$$\tilde{\Psi}_{5n}\, p_m - \tilde{\Psi}_{5m}\, p_n = \bar{\bar{\mu}}_4\, c\, \tilde{\Psi}_{mn} + i\, (\tilde{\Psi}\, \bar{\bar{\pi}})_{mn} \qquad (m, n = 1, 2, 3, 4) \tag{183}$$

and for $m = 5$:

$$\sum_{k=1}^{4} \tilde{\Psi}_{kn}\, p_k = -\bar{\bar{\mu}}_5\, c\, \tilde{\Psi}_{5n} - i\, (\tilde{\Psi}\, \bar{\bar{\pi}})_{5n} \qquad (n = 1, 2, 3, 4). \tag{184}$$

After multiplying equ. (184) by p_n from the right and summing over n from 1 to 4 we get:

$$\sum_{n=1}^{4} \tilde{\Psi}_{5n}\, p_n = -\frac{i}{\bar{\bar{\mu}}_5\, c} \sum_{n=1}^{4} (\tilde{\Psi}\, \bar{\bar{\pi}})_{5n}\, p_n. \tag{185}$$

If we insert

$$\tilde{\Psi}_{kn} = \frac{1}{\bar{\bar{\mu}}_4 c} (\tilde{\Psi}_{5n} p_k - \tilde{\Psi}_{5k} p_n) - \frac{i}{\bar{\bar{\mu}}_4 c} (\tilde{\Psi} \bar{\bar{\pi}})_{kn} \qquad (k, n = 1, 2, 3, 4) \tag{186}$$

in equ. (184) we obtain by equ. (185):

$$\tilde{\Psi}_{5n} \left(\sum_{k=1}^{4} p_k^2 + \bar{\bar{\mu}}_4 \bar{\bar{\mu}}_5 c^2 \right) = - i \bar{\bar{\mu}}_4 c (\tilde{\Psi} \bar{\bar{\pi}})_{5n} + i \sum_{k=1}^{4} (\tilde{\Psi} \pi)_{kn} p_k -$$

$$- \frac{i}{\bar{\bar{\mu}}_5 c} \sum_{k=1}^{4} (\tilde{\Psi} \bar{\bar{\pi}})_{5k} p_k p_n \qquad (n = 1, 2, 3, 4). \tag{187}$$

By eqs. (177), (178), (23) and (24) we obtain:

$$\left(\sum_{k=1}^{4} \frac{\partial^2}{\partial x_k^2} - \varkappa^2 \right) \tilde{\Psi}_{5n} = \frac{i \bar{\mu}_4 c}{\hbar^2} (\tilde{\Psi} \bar{\bar{\pi}})_{5n} +$$

$$+ \frac{1}{\hbar} \sum_{k=1}^{4} \frac{\partial}{\partial x_k} (\tilde{\Psi} \bar{\bar{\pi}})_{kn} - \frac{i}{\bar{\bar{\mu}}_5 c} \frac{\partial}{\partial x_n} \sum_{k=1}^{4} \frac{\partial}{\partial x_k} (\tilde{\Psi} \bar{\bar{\pi}})_{5k} \qquad (n = 1, 2, 3, 4) \tag{188}$$

and for the other components:

$$\tilde{\Psi}_{kn} = \frac{i \hbar}{\mu_4 c} \left(\frac{\partial}{\partial x_k} \tilde{\Psi}_{5n} - \frac{\partial}{\partial x_n} \tilde{\Psi}_{5k} \right) - \frac{i}{\mu_4 c} (\tilde{\Psi} \pi)_{kn} \qquad (k, n = 1, 2, 3, 4) \tag{189}$$

and

$$\tilde{\Psi}_{54} = \frac{i \hbar}{\bar{\bar{\mu}}_5 c} \sum_{k=1}^{4} \frac{\partial}{\partial x_k} \tilde{\Psi}_{4k} - \frac{i}{\bar{\bar{\mu}}_5 c} (\tilde{\Psi} \bar{\bar{\pi}})_{54}. \tag{190}$$

With the help of eqs. (33), (151) and (34) it is easily shown that eqs. (188), (189) and (190) are the conjugate complex of eqs. (179), (180) and (181).

Finally we consider the possibility of assuming Ψ to be transformed as a 5-dimensional antisymmetrical tensor of the 3rd rank, i. e. a 5-dimensional pseudotensor Ψ_{kl} of the 2nd rank. For transformations in the 4-dimensional subspace ($\varepsilon_{k5} = 0$) this means that Ψ_{kl} ($k, l = 1, 2, 3, 4$) is transformed like a 4-dimensional pseudotensor of the 2nd rank and Ψ_{5k} ($k = 1, 2, 3, 4$) is transformed like a 4-pseudovector. The transformation formula of the pseudotensor Ψ_{kl} is for infinitesimal transformations exactly the same as if it were an ordinary tensor, i. e. equ. (129) and the subsequent deduction of the matrix representation of the $\bar{\beta}_k$:s are valid for this case too. This implies the well-known fact that even in the pseudotensor case the β_k:s give the proper description of infinitesimal transformations when used in eqs. (21) and (6).

For the spatial reflexion of equ. (38) we have in this case to put

$$Q = - \bar{\beta}.$$

CHAPTER V

Formulation of the meson pair theory

We assume that the nucleons obey the ordinary DIRAC equation, and that they interact with a meson field, obeying the generalized DIRAC equations, considered in Chapters III and IV. When the treatment in the present chapter is applicable to both $\bar{\beta}_k$ and $\bar{\bar{\beta}}_k$, we will call these matrices β_k for short. The equations of the meson pair theory are then:

$$\left(\sum_{k=1}^{4} \gamma_k p_k - \vartheta_\gamma - \pi_\gamma\right) \psi = 0 \tag{191}$$

and

$$\left(\sum_{k=1}^{4} \beta_k p_k - \vartheta_\beta - \pi_\beta\right) \Psi = 0, \tag{192}$$

where ϑ_γ and ϑ_β are given by eqs. (56) and (108) or (165) respectively, and where π_γ and π_β by eqs. (8), (34) and (42) have to fulfil the following conditions:

$$Q_\gamma^{-1} \pi'_\gamma Q_\gamma = \pi_\gamma, \tag{193}$$

$$Q_\beta^{-1} \pi'_\beta Q_\beta = \pi_\beta, \tag{194}$$

$$\gamma_4 \pi_\gamma = - \pi_\gamma^* \gamma_4, \tag{195}$$

$$\beta \pi_\beta = - \pi_\beta^* \beta, \tag{196}$$

$$\tilde{\psi} (\pi_\gamma \psi) = (\tilde{\psi} \pi_\gamma) \psi, \tag{197}$$

$$\tilde{\Psi} (\pi_\beta \Psi) = (\tilde{\Psi} \pi_\beta) \Psi. \tag{198}$$

Eqs. (193) and (194) have to be fulfilled to ensure the LORENTZ invariance of the DIRAC equations. Eqs. (195) and (196) give an expression for the Hermiticity of the interaction. Eqs. (197) and (198) have to be fulfilled if we want the 4-divergence of the current 4-vector to vanish for the nucleons and for the mesons separately, i. e. if we want the total charge of the nucleons and the total charge of the mesons to be conserved separately.

Since the mean value of a matrix is the same, no matter whether the matrix operates to the left or to the right, eqs. (197) and (198) are seen to be fulfilled if we assume that π_γ contains only the γ_k:s and expressions of the type $\tilde{\Psi} \beta_k \beta_l \cdots \Psi$ and π_β contains only the β_k:s and expressions of the type $\tilde{\psi} \gamma_k \gamma_l \cdots \psi$. The validity of eqs. (197) and (198) excludes for example the possibility of using the p_k:s in the π:s, since a differential operator gives different results when operating to the left and to the right without integration over the whole space. The most general form of π_γ and π_β is thus

$$\pi_\gamma = \tilde{\Psi} I (\gamma_k, \gamma_l, \cdots \beta_m, \beta_n, \cdots) \Psi \tag{199}$$

and

$$\pi_\beta = \tilde{\psi} I (\gamma_k, \gamma_l, \cdots \beta_m, \beta_n, \cdots) \psi, \tag{200}$$

where the function I has to be the same for both π_γ and π_β, if we assume it possible to derive the two eqs. (191) and (192) from the same Hamiltonian.

The conditions of eqs. (197) and (198) being now automatically fulfilled, we have to determine the conditions which must be imposed on I if π_β and π_γ are to satisfy eqs. (193), (194), (195) and (196).

The most general form of I is clearly a sum of products of the type $\gamma_k \gamma_l \cdots \cdots \beta_m \beta_n \cdots$, so that π_γ is a sum of products of the type $\gamma_k \gamma_l \cdots \tilde{\Psi} \beta_m \beta_n \cdots \Psi$ and π_β a sum of products of the type $\beta_m \beta_n \cdots \tilde{\psi} \gamma_k \gamma_l \cdots \psi$. Let us now consider, for example,

$$\pi_\beta = \sum c_i \cdot \beta_m \beta_n \cdots \tilde{\psi} \gamma_k \gamma_l \cdots \psi. \tag{201}$$

We know from equ. (39) that under a LORENTZ transformation $\tilde{\psi} \gamma_k \gamma_l \cdots \psi$ is transformed as a tensor, i. e.

$$\pi'_\beta = \sum c_i \sum_{r=1}^{4} \sum_{s=1}^{4} \cdots a_{kr} a_{ls} \cdots \beta_m \beta_n \cdots \tilde{\psi} \gamma_r \gamma_s \cdots \psi. \tag{202}$$

Let us now form $Q_\beta^{-1} \pi'_\beta Q_\beta$ and make use of equ. (9):

$$\begin{aligned} Q_\beta^{-1} \pi'_\beta Q_\beta &= \sum c_i \sum_{r=1}^{4} \sum_{s=1}^{4} \cdots a_{kr} a_{ls} \cdots Q_\beta^{-1} \beta_m Q_\beta Q_\beta^{-1} \beta_n Q_\beta \cdots \tilde{\psi} \gamma_r \gamma_s \cdots \psi = \\ &= \sum c_i \sum_{r=1}^{4} \sum_{s=1}^{4} \cdots \sum_{p=1}^{4} \sum_{t=1}^{4} \cdots a_{kr} a_{ls} \cdots a_{mp} a_{nt} \cdots \beta_p \beta_t \cdots \tilde{\psi} \gamma_r \gamma_s \cdots \psi. \end{aligned} \tag{203}$$

This expression must by equ. (194) be identical with the π_β of equ. (201), and this is clearly the case if

$$\begin{aligned} &\sum c_i \beta_m \beta_n \cdots \gamma_k \gamma_l \cdots = \\ &= \sum c_i \sum_{r=1}^{4} \sum_{s=1}^{4} \cdots \sum_{p=1}^{4} \sum_{t=1}^{4} \cdots a_{kr} a_{ls} \cdots a_{mp} a_{nt} \cdots \beta_p \beta_t \cdots \gamma_r \gamma_s \cdots. \end{aligned} \tag{204}$$

Hence the condition for the validity of equ. (194) is that $I = \Sigma c_i \beta_m \beta_n \cdots \gamma_k \gamma_l \cdots$ has the form of an invariant, if the β_k:s and γ_k:s are considered as vector components. Clearly this property is the condition also for the validity of equ. (193).

We have now only to investigate eqs. (195) and (196). E. g. for π_β equ. (196) means that the quantity

$$\beta \tilde{\psi} I \psi = \psi^* \beta \gamma_4 I \psi \tag{205}$$

is anti-Hermitian, or that $\beta \gamma_4 I$ is anti-Hermitian, i. e.

$$(\beta \gamma_4 I)^* = - \beta \gamma_4 I. \tag{206}$$

Obviously we get the same condition for the validity of equ. (195). Summing up, we have found that all the conditions of eqs. (193) to (198) are fulfilled if:

1) the I of eqs. (199) and (200) is a function only of the β_k:s and γ_k:s, having the form of an invariant if the β_k:s and γ_k:s are considered as vector-components;
2) $\beta \gamma_4 I$ is anti-Hermitian.

The most general expression of I having the property 1) is clearly a sum of expressions of the type

$$\cdots \sum_{k=1}^{4} \cdots \sum_{l=1}^{4} \cdots \sum_{m=1}^{4} \cdots \gamma_k \cdots \gamma_l \cdots \gamma_k \cdots \beta_m \cdots \beta_m \cdots \beta_l \cdots, \tag{207}$$

where the summations are always taken over pairs of indices, but where the order of the factors are quite arbitrary.

Firstly, if a summation goes over a pair of γ_k:s as the γ_k:s in the expression (207), we can always by means of equ. (46) move one of them close to the other, and performing the summations over all the δ-symbols appearing, we can thus reduce the expression (207) to a sum of expressions of the type

$$\cdots \sum_{k=1}^{4} \cdots \sum_{l=1}^{4} \cdots \gamma_k \cdots \beta_l \cdots \beta_k \cdots \beta_l \cdots, \tag{208}$$

where no summations over pairs of γ_k:s occur.

By means of HARISH-CHANDRA's procedure for reducing the general invariant in the β_k:s, reproduced in Chapter III, we can now move such β_k:s, as are summed together with another β_k, out to the left, thus reducing the expression (208) to a sum of expressions of the type

$$\cdots \sum_{k=1}^{4} \cdots \sum_{l=1}^{4} \cdots \gamma_k \cdots \gamma_l \cdots (c_4 \omega_4 + c_5 \omega_5) \cdots \beta_k \cdots \beta_l \cdots, \tag{209}$$

where c_4 and c_5 are ordinary numbers, and no summations over pairs of β_k:s or γ_k:s occur.

We will now proceed to prove that in the expression (209) we can confine ourselves only to such terms in which all indices $k, l, \cdots$ take different values, since all terms in which two or more indices take the same value can be reduced to expressions of the same type as (209) but with fewer β_k:s.

To prove this we note that the sum of all the terms of (209), in which two or more indices take the same value, can be written

$$\sum_{k=1}^{4} \sum_{l=1}^{4} \sum_{m=1}^{4} \cdots [1 - (1 - \delta_{kl})(1 - \delta_{km})(1 - \delta_{lm}) \cdots] \cdot$$
$$\cdots \gamma_k \cdots \gamma_l \cdots \gamma_m \cdots (c_4 \omega_4 + c_5 \omega_5) \cdots \beta_k \cdots \beta_l \cdots \beta_m \cdots, \tag{210}$$

where δ-symbols of all possible index-pairs occur in the bracket.

If we expand the bracket, any term in the resulting sum is of the type:

$$\cdots \sum_{k=1}^{4} \cdots \sum_{l=1}^{4} \cdots \sum_{m=1}^{4} \cdots \sum_{n=1}^{4} \cdots \delta_{kl} \cdots \delta_{mn} \cdots \gamma_k \cdots \gamma_l \cdots \gamma_m \cdots \gamma_n \cdots \cdot$$
$$\cdot (c_4\,\omega_4 + c_5\,\omega_5) \cdots \beta_k \cdots \beta_l \cdots \beta_m \cdots \beta_n \cdots, \qquad (211)$$

where at least one δ-symbol occurs.

After summation over one index in each δ-symbol we get

$$\cdots \sum_{r=1}^{4} \cdots \gamma_r \cdots \gamma_r \cdots \gamma_r \cdots (c_4\,\omega_4 + c_5\,\omega_5) \cdots \beta_r \cdots \beta_r \cdots \beta_r \cdots, \qquad (212)$$

where we have written out only one group of equal indices, over which we have to sum. Using equ. (46) we can reduce this expression to a sum of terms either of the type

$$\cdots \sum_{r=1}^{4} \cdots \gamma_r \cdots (c_4\,\omega_4 + c_5\,\omega_5) \cdots \beta_r \cdots \beta_r \cdots \beta_r \cdots \qquad (213)$$

with one γ_r and an odd number of β_r:s, or of the type

$$\cdots \sum_{r=1}^{4} \cdots \gamma_k \cdots (c_4\,\omega_4 + c_5\,\omega_5) \cdots \beta_r \cdots \beta_r \cdots \beta_r \cdots \beta_r \cdots \qquad (214)$$

with no γ_r and an even number of β_r:s.

Using equ. (88) we can now move the β_r:s out to the left, giving in the first case only one β_r and in the other case $\sum_{r=1}^{4} \beta_r\,\beta_r$, which sum is of the type $(c_4'\,\omega_4 + c_5'\,\omega_5)$ and can thus be incorporated with $(c_4\,\omega_4 + c_5\,\omega_5)$. Every time we move a β_r we get of course in addition terms from the left hand side of equ. (88), which after summation over one of the indices of the δ-symbol are again of the type (212) but with fewer β_k:s. We proceed in the same manner with all other groups of equal indices and with all terms appearing with fewer β_k:s. Finally we have thus reduced the expression (210) to a sum of terms of the type (209), but with fewer β_k:s than (210). With the terms of these expressions, in which two or more indices take the same value, we repeat the procedure described, and finally reduce the expression (210) to a sum of expressions (209) with all indices different and with fewer β_k:s than we had from the beginning.

We have thus proved that the most general I can be expressed as a sum of all the possible expressions of the type (209) with all indices different. Such summations will be denoted below by Σ', so that the expressions of which I has to be a linear combination are:

$$\cdots \sum_{k=1}^{4}{}' \cdots \sum_{l=1}^{4}{}' \cdots \gamma_k \cdots \gamma_l \cdots (c_4\,\omega_4 + c_5\,\omega_5) \cdots \beta_k \cdots \beta_l \cdots. \qquad (215)$$

Since the indices only go from 1 to 4 no such expressions with more than four γ_k:s and four β_k:s exist.

Before we list the possible expressions (215) we will prove the following formulas:

$$\omega_5 \beta_l \beta_k \cdots = \beta_l \beta_k \cdots \omega_5 \text{ (even number of } \beta_k\text{:s)}, \tag{216}$$

$$\omega_4 \beta_l \beta_k \cdots = \beta_l \beta_k \cdots \omega_4 \text{ (even number of } \beta_k\text{:s)}, \tag{217}$$

$$\omega_4 \beta_l \beta_k \beta_m \cdots = \beta_l \beta_k \beta_m \cdots \omega_5 \text{ (odd number of } \beta_k\text{:s)}, \tag{218}$$

$$\omega_5 \beta_l \beta_k \beta_m \cdots = \beta_l \beta_k \beta_m \cdots \omega_4 \text{ (odd number of } \beta_k\text{:s)}. \tag{219}$$

It is easily seen that they are the immediate consequence of the simplest one, i. e.

$$\omega_5 \beta_l = \beta_l \omega_4 \tag{220}$$

and its Hermitian conjugate:

$$\omega_4 \beta_l = \beta_l \omega_5. \tag{221}$$

These two equations can be easily inferred from HARISH-CHANDRA's »Γ-formalism» [11], but can be more directly proved in the following way. We write

$$\omega_5 = a + b \sum_{k=1}^{4} \beta_k \beta_k \tag{222}$$

and

$$\omega_4 = 1 - a - b \sum_{k=1}^{4} \beta_k \beta_k . \tag{223}$$

The expression, which we have to prove to be zero, is:

$$\begin{aligned}
&\omega_5 \beta_l - \beta_l \omega_4 = \\
&= \left(a + b \sum_{k=1}^{4} \beta_k \beta_k\right) \beta_l - \beta_l \left(1 - a - b \sum_{k=1}^{4} \beta_k \beta_k\right) = \\
&= (2a - 1) \beta_l + b \sum_{k=1}^{4} (\beta_k \beta_k \beta_l + \beta_l \beta_k \beta_k) = \\
&= (2a - 1) \beta_l + b \sum_{k=1}^{4} (\beta_k \delta_{kl} + \beta_l) = \\
&= \beta_l (2a - 1 + 5b).
\end{aligned} \tag{224}$$

Now equ. (104) for $\bar{\omega}_5$ gives $\bar{a} = -\frac{1}{3}$ and $\bar{b} = \frac{1}{3}$, and equ. (161) for ω_5 gives $\bar{\bar{a}} = -2$ and $\bar{\bar{b}} = 1$, and in both cases we see, that $2a - 1 + 5b$ vanishes, and the formula (220) is thus proved.

The possible expressions (215) for I are now easily seen to be

$$i\,\omega_5, \tag{225}$$

$$i\,\omega_4, \tag{226}$$

$$i\sum_{k=1}^{4}\gamma_k(\omega_5\beta_k+\beta_k\omega_5), \tag{227}$$

$$i\sum_{k=1}^{4}\gamma_k(\omega_4\beta_k+\beta_k\omega_4), \tag{228}$$

$$\sum_{k=1}^{4}\gamma_k(\omega_5\beta_k-\beta_k\omega_5), \tag{229}$$

$$\sum_{k=1}^{4}\gamma_k(\omega_4\beta_k-\beta_k\omega_4), \tag{230}$$

$$\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\gamma_k\gamma_l(\omega_5\beta_k\beta_l+\beta_l\beta_k\omega_5), \tag{231}$$

$$\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\gamma_k\gamma_l(\omega_4\beta_k\beta_l+\beta_l\beta_k\omega_4), \tag{232}$$

$$i\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\gamma_k\gamma_l(\omega_5\beta_k\beta_l-\beta_l\beta_k\omega_5), \tag{233}$$

$$i\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\gamma_k\gamma_l(\omega_4\beta_k\beta_l-\beta_l\beta_k\omega_4), \tag{234}$$

$$\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\sum_{m=1}^{4}{}'\gamma_k\gamma_l\gamma_m(\omega_5\beta_k\beta_l\beta_m+\beta_m\beta_l\beta_k\omega_5), \tag{235}$$

$$\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\sum_{m=1}^{4}{}'\gamma_k\gamma_l\gamma_m(\omega_4\beta_k\beta_l\beta_m+\beta_m\beta_l\beta_k\omega_4), \tag{236}$$

$$i\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\sum_{m=1}^{4}{}'\gamma_k\gamma_l\gamma_m(\omega_5\beta_k\beta_l\beta_m-\beta_m\beta_l\beta_k\omega_5), \tag{237}$$

$$i\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\sum_{m=1}^{4}{}'\gamma_k\gamma_l\gamma_m(\omega_4\beta_k\beta_l\beta_m-\beta_m\beta_l\beta_k\omega_4), \tag{238}$$

$$i\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\sum_{m=1}^{4}{}'\sum_{n=1}^{4}{}'\gamma_k\gamma_l\gamma_m\gamma_n(\omega_5\beta_k\beta_l\beta_m\beta_n+\beta_n\beta_m\beta_l\beta_k\omega_5), \tag{239}$$

$$i\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\sum_{m=1}^{4}{}'\sum_{n=1}^{4}{}'\gamma_k\gamma_l\gamma_m\gamma_n(\omega_4\beta_k\beta_l\beta_m\beta_n+\beta_n\beta_m\beta_l\beta_k\omega_4), \tag{240}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \gamma_k \gamma_l \gamma_m \gamma_n (\omega_5 \beta_k \beta_l \beta_m \beta_n - \beta_n \beta_m \beta_l \beta_k \omega_5), \tag{241}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \gamma_k \gamma_l \gamma_m \gamma_n (\omega_4 \beta_k \beta_l \beta_m \beta_n - \beta_n \beta_m \beta_l \beta_k \omega_4), \tag{242}$$

where the second terms in the brackets and the i:s are inserted to make $\gamma_4 \beta$ times the expressions anti-Hermitian.

Now it is easy to see by means of eqs. (216)—(219), that (227) and (228) are equal, (229) and (230) equal with opposite sign, (231) and (232) vanish, (235) and (236) are equal with opposite sign, (237) and (238) are equal, and (241) and (242) vanish.

Moreover, the expressions (239) and (240), are either zero or proportional to the expression

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \gamma_k \gamma_l \gamma_m \gamma_n \beta_k \beta_l \beta_m \beta_n, \tag{243}$$

as can be proved in the following way. It is easily seen that both can be written as

$$\left(a + b \sum_{r=1}^{4} \beta_r \beta_r\right) \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \gamma_k \gamma_l \gamma_m \gamma_n \beta_k \beta_l \beta_m \beta_n . \tag{244}$$

Now the index r takes on summation the four different values k, l, m, n, and the expression is thus

$$\begin{aligned} &\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \gamma_k \gamma_l \gamma_m \gamma_n [a \beta_k \beta_l \beta_m \beta_n + b \beta_k \beta_k \beta_k \beta_l \beta_m \beta_n + \\ &+ b \beta_l \beta_l \beta_k \beta_l \beta_m \beta_n + b \beta_m \beta_m \beta_k \beta_l \beta_m \beta_n + b \beta_n \beta_n \beta_k \beta_l \beta_m \beta_n] = \\ &= \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \gamma_k \gamma_l \gamma_m \gamma_n (a + 2 b) \beta_k \beta_l \beta_m \beta_n . \end{aligned} \tag{245}$$

For vector mesons, i. e. for the β_k:s, all the terms (225)—(242) are possible except those which we just have found to vanish. The possible I:s in this case are thus:

$$i \bar{\bar{\omega}}_5 , \tag{246}$$

$$i \bar{\bar{\omega}}_4 , \tag{247}$$

$$i \sum_{k=1}^{4} \gamma_k \bar{\bar{\beta}}_k , \tag{248}$$

$$\sum_{k=1}^{4} \gamma_k (\bar{\bar{\omega}}_5 \beta_k - \beta_k \omega_5), \tag{249}$$

$$i \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \gamma_k \gamma_l \bar{\omega}_5 \bar{\beta}_k \bar{\beta}_l , \tag{250}$$

$$i \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \gamma_k \gamma_l \bar{\omega}_4 \bar{\beta}_k \bar{\beta}_l , \tag{251}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \gamma_k \gamma_l \gamma_m (\bar{\omega}_5 \bar{\beta}_k \bar{\beta}_l \bar{\beta}_m + \bar{\beta}_m \bar{\beta}_l \bar{\beta}_k \bar{\omega}_5), \tag{252}$$

$$i \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \gamma_k \gamma_l \gamma_m \bar{\beta}_k \bar{\beta}_l \bar{\beta}_m , \tag{253}$$

$$i \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \gamma_k \gamma_l \gamma_m \gamma_n \bar{\beta}_k \bar{\beta}_l \bar{\beta}_m \bar{\beta}_n , \tag{254}$$

where the expressions (227), (233), (234) and (237) are transformed into (248), (250), (251) and (253), respectively.

For scalar mesons, terms with more than two $\bar{\beta}_k$:s vanish by equ. (85).

Of the remaining terms, the ones corresponding to (250) and (251) are either zero or proportional to

$$i \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \gamma_k \gamma_l \bar{\beta}_k \bar{\beta}_l , \tag{255}$$

as can be proved in writing them in the following way:

$$\begin{aligned} \left(a + b \sum_{r=1}^{4} \beta_r \bar{\beta}_r\right) \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \gamma_k \gamma_l \bar{\beta}_k \bar{\beta}_l = \\ = a \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \gamma_k \gamma_l \bar{\beta}_k \beta_l + b \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \gamma_k \gamma_l (\bar{\beta}_k \bar{\beta}_k \bar{\beta}_k \beta_l + \beta_l \beta_l \beta_k \beta_l) = \\ = (a + b) \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \gamma_k \gamma_l \bar{\beta}_k \bar{\beta}_l . \end{aligned} \tag{256}$$

The possible I:s for the scalar mesons are thus:

$$i \bar{\omega}_5 , \tag{257}$$

$$i \bar{\omega}_4 , \tag{258}$$

$$i \sum_{k=1}^{4} \gamma_k \beta_k , \tag{259}$$

$$\sum_{k=1}^{4} \gamma_k (\omega_5 \beta_k - \beta_k \bar{\omega}_5), \tag{260}$$

$$i \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \gamma_k \gamma_l \bar{\beta}_k \bar{\beta}_l . \tag{261}$$

We want to draw attention to the fact that the interactions listed here are valid not only in the scalar and vector but also in the pseudoscalar and pseudovector theories. This is due to the fact that the β_k:s of the scalar and vector theories are equally valid in the pseudoscalar and pseudovector theories respectively, making in both cases eqs. (191) and (192) invariant not only for proper LORENTZ transformations but also for spatial reflexions. The pair theories thus give, in contrast to ordinary meson theories, no different interactions for the pseudofields.

We are now able to prove that, if μ_4 and μ_5 are both different from zero, there is no lack in generality in assuming them to have the same value.

We will prove this by transforming the two equations

$$\left(\sum_{k=1}^{4} \gamma_k p_k + i c M - \tilde{\Psi} I \Psi\right) \psi = 0 \tag{262}$$

and

$$\left(\sum_{k=1}^{4} \beta_k p_k + i c (\mu_4 \omega_4 + \mu_5 \omega_5) - \tilde{\psi} I \psi\right) \Psi = 0 \tag{263}$$

to a form, in which the mass term of the second equation is a constant times the unit matrix.

Let us for this reason divide I into three parts

$$I = I_0 + I_4 + I_5, \tag{264}$$

where I_0 stands for the terms (248), (249), (252) and (253) with an odd number of β_k:s, I_4 for the terms (247) and (251) with an even number of β_k:s and an ω_4-factor, and I_5 finally for the terms (246) and (250) with an even number of β_k:s and an ω_5-factor. As is seen from eqs. (245), (256), (161) and (104) the terms (254) and (261) are to be included in I_4.

Let us furthermore introduce the new wave function

$$\Psi' = (\mu_4 \mu_5)^{-\frac{1}{4}} (\sqrt{\mu_4}\, \omega_4 + \sqrt{\mu_5}\, \omega_5) \Psi,$$

giving

$$\Psi = (\mu_4 \mu_5)^{\frac{1}{4}} \left(\frac{\omega_4}{\sqrt{\mu_4}} + \frac{\omega_5}{\sqrt{\mu_5}}\right) \Psi'.$$

By taking the Hermitian conjugate of the last two equations and multiplying them from the right by β, we obtain, since β commutes with both ω_4 and ω_5:

$$\tilde{\Psi}' = \tilde{\Psi} (\sqrt{\mu_4}\, \omega_4 + \sqrt{\mu_5}\, \omega_5) (\mu_4 \mu_5)^{-\frac{1}{4}}$$

and

$$\tilde{\Psi} = \tilde{\Psi}' \left(\frac{\omega_4}{\sqrt{\mu_4}} + \frac{\omega_5}{\sqrt{\mu_5}}\right) (\mu_4 \mu_5)^{\frac{1}{4}}.$$

Equ. (262) now gives:

$$\left\{ \sum_{k=1}^{4} \gamma_k p_k + i c M - \tilde{\Psi}' \left(\frac{\omega_4}{\sqrt{\mu_4}} + \frac{\omega_5}{\sqrt{\mu_5}} \right) \cdot \right.$$
$$\left. \cdot \sqrt{\mu_4 \mu_5} \, (I_0 + I_4 + I_5) \left(\frac{\omega_4}{\sqrt{\mu_4}} + \frac{\omega_5}{\sqrt{\mu_5}} \right) \Psi' \right\} \psi = 0$$

or by eqs. (216)—(219):

$$\left\{ \sum_{k=1}^{4} \gamma_k p_k + i c M - \tilde{\Psi}' \left(I_0 + \sqrt{\frac{\mu_5}{\mu_4}} \, I_4 + \sqrt{\frac{\mu_4}{\mu_5}} \, I_5 \right) \Psi' \right\} \psi = 0. \tag{265}$$

If, on the other hand, we multiply equ. (263) from the left by

$$(\mu_4 \mu_5)^{-\frac{1}{4}} \left(\sqrt{\mu_4} \, \omega_5 + \sqrt{\mu_5} \, \omega_4 \right)$$

we obtain for the first term:

$$\sum_{k=1}^{4} \beta_k p_k (\mu_4 \mu_5)^{-\frac{1}{4}} \left(\sqrt{\mu_4} \, \omega_4 + \sqrt{\mu_5} \, \omega_5 \right) \Psi = \sum_{k=1}^{4} \beta_k p_k \Psi'. \tag{266}$$

The mass term gives:

$$\begin{aligned} & i c \, (\mu_4 \mu_5)^{-\frac{1}{4}} \left(\sqrt{\mu_4} \, \omega_5 + \sqrt{\mu_5} \, \omega_4 \right) (\mu_4 \omega_4 + \mu_5 \omega_5) \Psi = \\ & = i c \left(\sqrt{\mu_4} \, \omega_5 + \sqrt{\mu_5} \, \omega_4 \right) \left(\sqrt{\mu_4} \, \omega_4 + \sqrt{\mu_5} \, \omega_5 \right) (\mu_4 \mu_5)^{-\frac{1}{4}} \left(\sqrt{\mu_4} \, \omega_4 + \sqrt{\mu_5} \, \omega_5 \right) \Psi = \\ & = i c \sqrt{\mu_4 \mu_5} \, \Psi'. \end{aligned} \tag{267}$$

By eqs. (218) and (219) we have for the I_0-term

$$- (\mu_4 \mu_5)^{-\frac{1}{4}} \left(\sqrt{\mu_4} \, \omega_5 + \sqrt{\mu_5} \, \omega_4 \right) \tilde{\psi} I_0 \psi \Psi = - \tilde{\psi} I_0 \psi \Psi', \tag{268}$$

for the I_4-term

$$\begin{aligned} & - (\mu_4 \mu_5)^{-\frac{1}{4}} \left(\sqrt{\mu_4} \, \omega_5 + \sqrt{\mu_5} \, \omega_4 \right) \tilde{\psi} I_4 \psi \Psi = - (\mu_4 \mu_5)^{-\frac{1}{4}} \sqrt{\mu_5} \, \tilde{\psi} I_4 \psi \Psi = \\ & = - \tilde{\psi} \sqrt{\frac{\mu_5}{\mu_4}} \, I_4 \psi \, (\mu_4 \mu_5)^{-\frac{1}{4}} \left(\sqrt{\mu_4} \, \omega_4 + \sqrt{\mu_5} \, \omega_5 \right) \Psi = - \tilde{\psi} \sqrt{\frac{\mu_5}{\mu_4}} \, I_4 \psi \Psi', \end{aligned} \tag{269}$$

and for the I_5-term

$$\begin{aligned} & - (\mu_4 \mu_5)^{-\frac{1}{4}} \left(\sqrt{\mu_4} \, \omega_5 + \sqrt{\mu_5} \, \omega_4 \right) \tilde{\psi} I_5 \psi \Psi = - (\mu_4 \mu_5)^{-\frac{1}{4}} \sqrt{\mu_4} \, \tilde{\psi} I_5 \psi \Psi = \\ & = - \tilde{\psi} \sqrt{\frac{\mu_4}{\mu_5}} \, I_5 \psi \, (\mu_4 \mu_5)^{-\frac{1}{4}} \left(\sqrt{\mu_4} \, \omega_4 + \sqrt{\mu_5} \, \omega_5 \right) \Psi = - \tilde{\psi} \sqrt{\frac{\mu_4}{\mu_5}} \, I_5 \psi \Psi'. \end{aligned} \tag{270}$$

If we add the expressions (266)—(270) we finally get:

$$\left\{\sum_{k=1}^{4}\beta_k p_k + i c\sqrt{\mu_4\mu_5} - \tilde{\psi}\left(I_0 + \sqrt{\frac{\mu_5}{\mu_4}}\, I_4 + \sqrt{\frac{\mu_4}{\mu_5}}\, I_5\right)\psi\right\}\Psi' = 0. \qquad (271)$$

Eqs. (262) and (263) are now transformed into eqs. (265) and (271), where the mass term of the second equation is a constant times the unit matrix and where the interaction term is a linear combination of the same terms as in the original equations but with different coefficients. Our statement is thus proved.

CHAPTER VI

The quantization of the meson field

The exact treatment of eqs. (191) and (192) would of course imply a quantization of both the nucleonic field ψ and the meson field Ψ. In the absence of methods for deriving useful results from such a general formalism, we will here confine ourselves to the usual approximate method of performing the quantization of the meson field under the assumption that the positions and spins of the nucleons are fixed. This approximation implies that the mean values $\tilde{\psi}\gamma_k\gamma_l\cdots\psi$ can be treated as ordinary numbers, which in the final result have to be substituted by

$$\tilde{\psi}\gamma_k\gamma_l\cdots\psi \to \sum_N \Omega_N\gamma_{4N}\gamma_{kN}\gamma_{lN}\cdots, \qquad (272)$$

where the summation over N is assumed to be performed over all the nucleons present and Ω_N is a source-function around the N:th nucleon.

By means of the meson equation

$$\left(\sum_{k=1}^{4}\beta_k p_k - \vartheta_\beta - \tilde{\psi} I \psi\right)\Psi = 0 \qquad (273)$$

we can now form the total Hamiltonian of the meson field:

$$H = \int \tilde{\Psi}\beta_4 E\,\Psi\, d\,V = -i c\int \tilde{\Psi}\beta_4 p_4\Psi\, d\,V =$$

$$= i c\int \tilde{\Psi}\left(\sum_{k=1}^{3}\beta_k p_k - \vartheta_\beta - \tilde{\psi} I \psi\right)\Psi\, d\,V. \qquad (274)$$

The method of finding the interaction potential between the nucleons consists in calculating the difference between the expression (274) and the same expression where the interaction term $\pi_\beta = \tilde{\psi} I \psi$ is cancelled. This difference, which gives the increase in the energy of the field due to the nuclear sources, is then used as the nuclear interaction potential in the Dirac equation for the nucleons.

In ordinary meson theories, as in the electromagnetic case, a first so-called »static» approximation of the interaction energy can be obtained by determining a static solution of the meson field equations, i. e. a solution of the equations, where all time derivatives have been cancelled. This static solution is then inserted in the Hamiltonian of the meson field, and the interaction energy of the nucleons is obtained as the difference between this Hamiltonian and the unperturbed one.

As has been kindly pointed out to the author by prof. FRÖHLICH, no such possibility exists in meson pair theory. In fact, if we cancel the time derivative p_4 in equ. (273), we get an equation, the solutions of which evidently give the result zero when inserted in the Hamiltonian (274). From the fact that no such »classical» approximation exists, we can conclude that the quantization of the meson field is an essential feature of the pair theories.

The quantization of the meson field can be performed in the following way:

Let us for short introduce the operator

$$\Theta = i\,c\left(\sum_{k=1}^{3} \beta_k\, p_k - \vartheta_\beta - \pi_\beta\right). \tag{275}$$

Then we can write eqs. (273) and (274) as:

$$\beta_4 E\,\Psi = \Theta\,\Psi \tag{276}$$

and

$$H = \int \tilde{\Psi}\beta_4 E\,\Psi\,d\,V = \int \tilde{\Psi}\,\Theta\,\Psi\,d\,V. \tag{277}$$

Let us now following KEMMER [18] define the momentum Π_r, conjugate to the component Ψ_r, as the r-component of

$$\Pi = \beta_4\,\Psi. \tag{278}$$

According to this definition there exist no conjugate momenta to the redundant components, for which we have $(\beta_4\,\Psi)_r = 0$. Thus, only Ψ_4 and Ψ_5 in the scalar meson case and Ψ_{41}, Ψ_{42}, Ψ_{43}, Ψ_{51}, Ψ_{52} and Ψ_{53} in the vector meson case have conjugate momenta. Similarly the conjugate momentum $\tilde{\Pi}_r$ to the component $\tilde{\Psi}_r$ is defined by the r-component of

$$\tilde{\Pi} = \tilde{\Psi}\,\beta_4. \tag{279}$$

The commutation rules are now:

$$[\Pi_r(\boldsymbol{r}'),\, \tilde{\Psi}_s(\boldsymbol{r}'')] = \delta(\boldsymbol{r}' - \boldsymbol{r}'')\,\delta_{rs} \qquad (r, s \text{ non-redundant}), \tag{280}$$

$$[\Psi_r(\boldsymbol{r}'),\, \tilde{\Pi}_s(\boldsymbol{r}'')] = \delta(\boldsymbol{r}' - \boldsymbol{r}'')\,\delta_{rs} \qquad (r, s \text{ non-redundant}), \tag{281}$$

and all other pairs of non-redundant components are assumed to commute.

From these commutation rules and HAMILTON's equations

$$i\hbar\frac{\partial\,\Psi_r}{\partial\,t} = [\Psi_r,\, H] \qquad (r \text{ non-redundant}), \tag{282}$$

$$i\hbar\frac{\partial \Pi_r}{\partial t} = [\Pi_r, H] \qquad (r \text{ non-redundant}), \tag{283}$$

we can now deduce the non-redundant components of equ. (276), if we assume this equation valid for the redundant indices, i. e.

$$(\Theta\,\Psi)_r = 0 \qquad (r \text{ redundant}). \tag{284}$$

In fact we have:

$$\begin{aligned}(\beta_4 E\,\Psi(\boldsymbol{r}'))_r &= i\hbar\frac{\partial \Pi_r(\boldsymbol{r}')}{\partial t} = [\Pi_r(\boldsymbol{r}'), H] = \\ &= [\Pi_r(\boldsymbol{r}'), \int \sum_{\substack{s\text{ non-}\\ \text{red.}}} \tilde{\Psi}_s(\boldsymbol{r}'')\,(\Theta\,\Psi(\boldsymbol{r}''))_s\,d\,V''] = \\ &= \int \sum_{\substack{s\text{ non-}\\ \text{red.}}} [\Pi_r(\boldsymbol{r}'), \tilde{\Psi}_s(\boldsymbol{r}'')]\,(\Theta\,\Psi(\boldsymbol{r}''))_s\,d\,V'' = \\ &= \int \sum_{\substack{s\text{ non-}\\ \text{red.}}} \delta_{rs}\,\delta(\boldsymbol{r}'-\boldsymbol{r}'')\,(\Theta\,\Psi(\boldsymbol{r}''))_s\,d\,V'' = (\Theta\,\Psi(\boldsymbol{r}'))_r.\end{aligned} \tag{285}$$

We will now expand the time dependent, quantized wave-functions Ψ, $\tilde{\Psi}$, Π and $\tilde{\Pi}$ in terms of the non-quantized eigenfunctions

$$\Psi^n(\boldsymbol{r})\cdot e^{-\frac{i}{\hbar}E_n t} \qquad \text{and} \qquad \tilde{\Psi}^n(\boldsymbol{r})\,e^{\frac{i}{\hbar}E_n t} \tag{286}$$

to the differential equation (276) and its adjoint equation. Here $\Psi^n(\boldsymbol{r})$ and $\tilde{\Psi}^n(\boldsymbol{r})$ are the solutions of the equation

$$\beta_4 E_n\,\Psi^n = \Theta\,\Psi^n \tag{287}$$

and its adjoint equation. These functions are orthogonal, since by multiplying equ. (287) from the left by $\tilde{\Psi}^m$ and multiplying the adjoint equation for $\tilde{\Psi}^m$ from the right by Ψ^n, subtracting and integrating we get:

$$(E_n - E_m)\int \tilde{\Psi}^m\,\beta_4\,\Psi^n\,d\,V = 0. \tag{288}$$

Further, we have by multiplying equ. (287) by $\tilde{\Psi}^n$ from the left:

$$E_n\int \tilde{\Psi}^n\,\beta_4\,\Psi^n\,d\,V = \int \tilde{\Psi}^n\,\Theta\,\Psi^n\,d\,V. \tag{289}$$

Now, as we will see in the next chapter, the right hand side of this equation is positive in the case when π_3 is zero. When $\pi_3 \neq 0$, this is not generally the case, but if π_3 is sufficiently small, i. e. if π_3 is a small perturbation of the meson field in free space, than we can still assume the right hand side of equ. (289) to be positive. We can thus normalize the Ψ^n:s according to

$$\int \tilde{\Pi}^m \Psi^n \, d\,V = \int \tilde{\Psi}^m \beta_4 \Psi^n \, d\,V = \delta_{mn} \cdot \delta_n , \tag{290}$$

where

$$\delta_n = \frac{E_n}{|E_n|} \tag{291}$$

is $+1$ for positive eigenvalues and -1 for negative eigenvalues.

Let us now assume the expansion

$$\Psi(\mathbf{r}) = \sum_n a_n \Psi^n(\mathbf{r}), \tag{292}$$

where the time-dependence of the eigenfunctions is taken into account by the time dependence of the a_n:s, which are thus supposed to be proportional to $e^{-\frac{i}{\hbar} E_n t}$.

Multiplying equ. (292) from the left by $\tilde{\Pi}^m(\mathbf{r})$ and integrating we get:

$$a_m = \delta_m \int \sum_s \tilde{\Pi}_s^m(\mathbf{r}') \, \Psi_s(\mathbf{r}') \, d\,V'. \tag{293}$$

We have further:

$$\tilde{\Psi}(\mathbf{r}) = \Psi^*(\mathbf{r}) \, \beta = \sum_n a_n^* \Psi^{n*}(\mathbf{r}) \, \beta = \sum_n a_n^* \tilde{\Psi}^n(\mathbf{r}). \tag{294}$$

By multiplying equ. (292) from the left and equ. (294) from the right by β_4 we obtain:

$$\Pi(\mathbf{r}) = \sum_n a_n \Pi^n(\mathbf{r}) \tag{295}$$

and

$$\tilde{\Pi}(\mathbf{r}) = \sum_n a_n^* \tilde{\Pi}^n(\mathbf{r}). \tag{296}$$

Since $\tilde{\Pi} = \tilde{\Psi} \beta_4 = \Psi^* \beta \beta_4$ we get from equ. (293):

$$a_n^* = \delta_n \int \sum_r \tilde{\Pi}_r(\mathbf{r}'') \, \Psi_r^n(\mathbf{r}'') \, d\,V''. \tag{297}$$

We are now able to determine the value of the expression:

$$[a_m, a_n^*] = \delta_n \, \delta_m \int \int \sum_r \sum_s \tilde{\Pi}_s^m(\mathbf{r}') \cdot \Psi_r^n(\mathbf{r}'') \, [\Psi_s(\mathbf{r}'), \tilde{\Pi}_r(\mathbf{r}'')] \, d\,V' \, d\,V'' =$$

$$= \delta_n \, \delta_m \int \sum_s \tilde{\Pi}_s^m(\mathbf{r}') \, \Psi_s^n(\mathbf{r}') \, d\,V', \tag{298}$$

by equ. (281). Equ. (290) gives finally:

$$[a_m, a_n^*] = \delta_m \, \delta_{mn}. \tag{299}$$

We will now express H as a function of the a_m:s. From eqs. (296) and (292) we have

$$\begin{aligned} H &= \int \tilde{\Pi} E \Psi \, d V = \\ &= \int \sum_m \sum_n a_m^* \tilde{\Pi}^m E_n a_n \Psi^n \, d V = \\ &= \sum_m \sum_n a_m^* a_n E_n \int \tilde{\Pi}^m \Psi^n \, d V = \sum_n a_n^* a_n \delta_n E_n . \end{aligned} \tag{300}$$

If we put

$$a_n = \xi_n \quad \text{if} \quad E_n = E_n^+ > 0, \tag{301}$$

$$a_n = \eta_n^* \quad \text{if} \quad E_n = E_n^- < 0, \tag{302}$$

we have from equ. (299) the ordinary oscillator relations:

$$[\xi_m, \xi_n^*] = [\eta_m, \eta_n^*] = \delta_{mn}, \tag{303}$$

and from equ. (300):

$$H = \sum_n E_n^+ \xi_n^* \xi_n - \sum_m E_m^- (\eta_m^* \eta_m + 1). \tag{304}$$

For the total charge we have:

$$\begin{aligned} Q &= e \int \tilde{\Psi} \beta_4 \Psi \, d V = e \int \tilde{\Pi} \Psi \, d V = \\ &= e \sum_n \sum_m a_n^* a_m \int \tilde{\Pi}^n \Psi^m \, d V = e \sum_n a_n^* a_n \delta_n . \end{aligned} \tag{305}$$

Inserting eqs. (301) and (302) we get:

$$Q = e \sum_n \xi_n^* \xi_n - e \sum_m (\eta_m^* \eta_m + 1). \tag{306}$$

As usual we interpret $\xi_n^* \xi_n$ as the number of positively charged mesons with the positive energy E_n^+ and $\eta_n^* \eta_n$ as the number of negatively charged mesons with the negative energy E_n^-.

The expressions (304) and (306) are, however, not satisfactory, as H is not symmetrical with respect to positive and negative energy eigenvalues, and Q does not vanish in the state of no mesons, i. e. when $\xi_n^* \xi_n = \eta_n^* \eta_n = 0$.

This insufficiency is, as is well known, due to the fact that we have used only one sign for the meson charge e in the original DIRAC equation for the meson, whereas the other sign is equally possible. To see this we introduce, as in MAJORANA's treatment of the electron [24], the electromagnetic field A_k in equ. (273):

$$\left(\sum_{k=1}^{4} \beta_k \left(p_k - \frac{e}{c} A_k \right) - \vartheta_\beta - \pi_\beta \right) \Psi = 0. \tag{307}$$

The interactions π_β are of two different types. Some of them, which we will denote by π'_β, do not change their value when the β_k-product, which they contain, is transposed, i. e. when rows and columns are exchanged. Denoting the transposition by a bar underneath we have

$$\underline{\pi'_\beta} = \pi'_\beta. \tag{308}$$

Making use of the fact that

$$\underline{a \cdot b \cdots c \cdot d} = \underline{d} \cdot \underline{c} \cdots \underline{b} \cdot \underline{a}, \tag{309}$$

and that in the representation of the β_k:s in Chapters III and IV

$$\underline{\beta_k} = -\beta_k, \tag{310}$$

we see that the interactions (246), (247), (249), (253) and (254) are of this first type.

The interactions of the other type, π''_β, change their sign under a transposition:

$$\underline{\pi''_\beta} = -\pi''_\beta. \tag{311}$$

Clearly the interactions (248), (250), (251) and (252) have this property.

We will assume that the interactions of the first kind are proportional to an even power and the interactions of the second kind to an odd power of the meson charge e.

We can thus write equ. (307):

$$\left(\sum_{k=1}^{4} \beta_k \left(p_k - \frac{e}{c} A_k\right) - \vartheta_\beta - e^{2m}\,\pi'_\beta - e^{2n-1}\,\pi''_\beta\right) \Psi = 0. \tag{312}$$

If we change e to $-e$ we get:

$$\left(\sum_{k=1}^{4} \beta_k \left(p_k + \frac{e}{c} A_k\right) - \vartheta_\beta - e^{2m}\,\pi'_\beta + e^{2n-1}\,\pi''_\beta\right) \Psi^L = 0, \tag{313}$$

where Ψ^L is the wave-function, charge conjugate to Ψ.

If we now transpose the adjoint equation of equ. (312), we get from eqs. (308), (310) and (311) and since p_k, which after the transposition operates to the right, has to be changed to $-p_k$:

$$\left(\sum_{k=1}^{4} -\beta_k \left(-p_k - \frac{e}{c} A_k\right) - \vartheta_\beta - e^{2m}\,\pi'_\beta + e^{2n-1}\,\pi''_\beta\right) \underline{\tilde{\Psi}} = 0. \tag{314}$$

The identification of eqs. (313) and (314) gives:

$$\Psi^L = \underline{\tilde{\Psi}}. \tag{315}$$

Transposition of equ. (312) and comparison with the adjoint equation of equ. (313) gives in the same way:

$$\tilde{\Psi}^L = \underline{\Psi}. \tag{316}$$

The identifications of Ψ^L with $\underline{\tilde{\Psi}}$ and $\tilde{\Psi}^L$ with $\underline{\Psi}$ are of course possible only if $\underline{\tilde{\Psi}}$ and $\underline{\Psi}$ are transformed under a LORENTZ transformation as Ψ and $\tilde{\Psi}$ respectively.

In fact, if we transpose eqs. (25) and (6) we get

$$\underline{\tilde{\Psi}}' = \underline{Q}^{-1}\, \underline{\tilde{\Psi}}$$

and

$$\underline{\Psi}' = \underline{\Psi}\, \underline{Q}.$$

For the proper LORENTZ transformations we have by eqs. (13) and (14)

$$\underline{Q} = 1 + \tfrac{1}{2} \sum_{k=1}^{4} \sum_{l=1}^{4} \varepsilon_{kl}\, \underline{q_{kl}}$$

$$Q^{-1} = 1 - \tfrac{1}{2} \sum_{k=1}^{4} \sum_{l=1}^{4} \varepsilon_{kl}\, \underline{q_{kl}}.$$

Since

$$q_{kl} = [\beta_k, \beta_l]$$

the q_{kl}:s are anti-Hermitian matrices with only real elements, and they consequently change their sign, when they are transposed. We have thus:

$$Q = 1 - \tfrac{1}{2} \sum_{k=1}^{4} \sum_{l=1}^{4} \varepsilon_{kl}\, q_{kl} = Q^{-1}$$

and

$$Q^{-1} = 1 + \tfrac{1}{2} \sum_{k=1}^{4} \sum_{l=1}^{4} \varepsilon_{kl}\, q_{kl} = Q.$$

For the spatial reflexions we have

$$Q^{-1} = Q = \pm\, \beta,$$

giving

$$Q^{-1} = Q = \underline{Q}^{-1} = \underline{Q}.$$

For both proper LORENTZ transformations and spatial reflexions we have consequently

$$\underline{\tilde{\Psi}}' = Q\, \underline{\tilde{\Psi}} \tag{317}$$

and

$$\underline{\Psi}' = \underline{\Psi}\, Q^{-1}, \tag{318}$$

in conformity with eqs. (6) and (25) respectively, and the identifications in eqs. (315) and (316) are thus justified.

The definition of the mean value a of any operator $a(e)$, which is an explicit function of e, is now according to MAJORANA:

$$a = \tfrac{1}{2}\left\{\int \tilde{\Psi} \beta_4 \, a(e) \, \Psi \, dV + \int \tilde{\Psi}^L \beta_4 \, a(-e) \, \Psi^L \, dV\right\}. \tag{319}$$

In eqs. (304) and (306) we have already given the values of the first term in the bracket if a is the total energy and the total charge.

The second term for the energy can be calculated if we make use of the transposed of eqs. (292) and (294), i. e.

$$\underline{\Psi} = \sum_n a_n \, \underline{\Psi}^n \tag{320}$$

and

$$\underline{\tilde{\Psi}} = \sum_m a_m^* \, \underline{\tilde{\Psi}}^m. \tag{321}$$

In fact we obtain:

$$\begin{aligned}\int \tilde{\Psi}^L \beta_4 E \, \Psi^L \, dV &= \int \underline{\Psi} \, \beta_4 E \, \underline{\tilde{\Psi}} \, dV = \\ &= -\sum_m \sum_n a_n a_m^* E_m \int \underline{\Psi}^n \beta_4 \, \underline{\tilde{\Psi}}^m \, dV = \\ &= -\sum_m \sum_n a_n a_m^* E_m \int \tilde{\Psi}^m \underline{\beta_4} \, \Psi^n \, dV = \\ &= \sum_n a_n a_n^* \delta_n E_n = \\ &= \sum_n E_n^+ (\xi_n^* \xi_n + 1) \quad \sum_m E_m^- \eta_m^* \eta_m .\end{aligned} \tag{322}$$

Eqs. (319), (304) and (322) now gives:

$$H = \sum_n E_n^+ (\xi_n^* \xi_n + \tfrac{1}{2}) - \sum_m E_m^- (\eta_m^* \eta_m + \tfrac{1}{2}), \tag{323}$$

which expression has the required symmetry between positive and negative eigenvalues. In the state of no mesons we have

$$H = \tfrac{1}{2} \sum_n E_n^+ - \tfrac{1}{2} \sum_m E_m^- . \tag{324}$$

For the second term in the bracket of equ. (319) for the total charge we have:

$$\begin{aligned}-e \int \tilde{\Psi}^L \beta_4 \, \Psi^L \, dV &= -e \int \underline{\Psi} \, \beta_4 \, \underline{\tilde{\Psi}} \, dV = \\ &= -e \sum_m \sum_n a_n a_m^* \int \underline{\Psi}^n \beta_4 \, \underline{\tilde{\Psi}}^m \, dV = \\ &= e \sum_n a_n a_n^* \delta_n = \\ &= e \sum_n (\xi_n^* \xi_n + 1) - e \sum_m \eta_m^* \eta_m .\end{aligned} \tag{325}$$

If there are as many positive as negative eigenvalues, eqs. (319), (306) and (325) give:

$$Q = e \sum_n \xi_n^* \xi_n - e \sum_m \eta_m^* \eta_m \,, \tag{326}$$

which has the required property of vanishing in the state of no mesons.

It follows immediately from eqs. (326) and (323), that the total meson charge Q has the required property of commuting with the total Hamiltonian H, since both Q and H are diagonal at the same time.

CHAPTER VII

Elimination of the redundant components in the scalar pair theory

We will calculate in this chapter the Hamiltonian (274), expressed only in the non redundant components Ψ_4 and Ψ_5, eliminating the redundant components by means of eqs. (114) and (124).

As was proved at the end of Chapter V, we can without lack of generality assume $\bar{\mu}_4 = \bar{\mu}_5 = \bar{\mu}$, and eqs. (114) and (124) are then:

$$\Psi_n = -\frac{1}{\bar{\mu}\, c} p_n \Psi_5 - \frac{i}{\bar{\mu}\, c} (\bar{\pi}_\beta \Psi)_n \qquad (n = 1, 2, 3), \tag{327}$$

$$\tilde{\Psi}_n = \frac{1}{\bar{\mu}\, c} \tilde{\Psi}_5 p_n - \frac{i}{\bar{\mu}\, c} (\tilde{\Psi} \bar{\pi}_\beta)_n \qquad (n = 1, 2, 3). \tag{328}$$

The Hamiltonian is:

$$\begin{aligned}
H &= i c \int \tilde{\Psi} \left(\sum_{k=1}^{3} \beta_k p_k + i \bar{\mu} c - \bar{\pi}_\beta \right) \Psi \, d V = \\
&= i c \int \Bigg[\sum_{k=1}^{3} \sum_{n=4}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_n \, i (\delta_{nk} \delta_{n'5} - \delta_{n5} \delta_{n'k}) p_k \Psi_{n'} + i \bar{\mu} c \tilde{\Psi}_4 \Psi_4 + \\
&\quad + i \bar{\mu} c \tilde{\Psi}_5 \Psi_5 - \sum_{n=4}^{5} \tilde{\Psi}_n (\bar{\pi}_\beta \Psi)_n \Bigg] d V = \\
&= \int \Bigg[c \sum_{k=1}^{3} \tilde{\Psi}_5 p_k \Psi_k - \bar{\mu} c^2 \tilde{\Psi}_4 \Psi_4 - \bar{\mu} c^2 \tilde{\Psi}_5 \Psi_5 - i c \sum_{n=4}^{5} \tilde{\Psi}_n (\bar{\pi}_\beta \Psi)_n \Bigg] d V = \\
&= \int \Bigg[-\frac{1}{\bar{\mu}} \sum_{k=1}^{3} \tilde{\Psi}_5 p_k^2 \Psi_5 - \frac{i}{\bar{\mu}} \sum_{k=1}^{3} \tilde{\Psi}_5 p_k (\bar{\pi}_\beta \Psi)_k - \bar{\mu} c^2 \tilde{\Psi}_4 \Psi_4 - \\
&\quad - \bar{\mu} c^2 \tilde{\Psi}_5 \Psi_5 - i c \sum_{n=4}^{5} \tilde{\Psi}_n (\bar{\pi}_\beta \Psi)_n \Bigg] d V ,
\end{aligned} \tag{329}$$

and finally by means of equ. (328):

$$H = \int \left[\frac{\hbar^2}{\bar{\mu}} \sum_{k=1}^{3} \frac{\partial \Psi_5^*}{\partial x_k} \cdot \frac{\partial \Psi_5}{\partial x_k} + \bar{\mu}\, c^2\, \Psi_4^*\, \Psi_4 + \bar{\mu}\, c^2\, \Psi_5^*\, \Psi_5 - \right.$$

$$\left. - i\, c\, \tilde{\Psi}\, \bar{\pi}_\beta\, \Psi + \frac{1}{\bar{\mu}} \sum_{k=1}^{3} (\tilde{\Psi}\, \bar{\pi}_\beta)_k\, (\bar{\pi}_\beta\, \Psi)_k \right] d\, V. \qquad (330)$$

If we put $\bar{\pi}_\beta = 0$ in this expression, we get the positive Hamiltonian for the scalar meson field in free space, and quantisation as in the preceding chapter is therefore possible, if we assume that the last two terms of equ. (330) are small in comparison with the first three.

The most general form of π we have found to be:

$$\begin{aligned} \pi_\beta = {} & i\, \bar{c}_1' \cdot \psi\, \psi\, \bar{\omega}_5 - \\ & - i\, \bar{c}_1''\, \tilde{\psi}\, \psi\, \bar{\omega}_4 - \\ & - i\, \bar{c}_2' \sum_{k=1}^{4} \tilde{\psi}\, \gamma_k\, \psi\, \bar{\beta}_k + \\ & + \bar{c}_2'' \sum_{k=1}^{4} \tilde{\psi}\, \gamma_k\, \psi\, (\bar{\omega}_5\, \bar{\beta}_k - \bar{\beta}_k\, \bar{\omega}_5) - \\ & - i\, \bar{c}_3 \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \tilde{\psi}\, \gamma_k\, \gamma_l\, \psi\, \bar{\beta}_k\, \bar{\beta}_l \end{aligned} \qquad (331)$$

where the $\bar{c}_k$:s are real, positive or negative constants.

Let us put

$$s_0 = \psi^*\, \psi, \qquad (332)$$

$$s_{0m} = \pm\, s_{0kl} = \psi^*\, \sigma_m\, \psi = \pm\, \psi^*\, \sigma_{kl}\, \psi, \qquad (333)$$

$$s_i = -\, \psi^*\, \varrho_i\, \psi, \qquad (334)$$

$$s_{im} = \pm\, s_{ikl} = -\, \psi^*\, \varrho_i\, \sigma_m\, \psi = \mp\, \psi^*\, \varrho_i\, \sigma_{kl}\, \psi, \qquad (335)$$

where $i, k, l, m = 1, 2, 3$ and where the upper signs are used if $k = 1$, $l = 2$, $m = 3$ or cyclic permutations, and the lower signs if this is not the case. s_0 and s_i are 3-dimensional scalars. s_{01}, s_{02}, s_{03} and s_{i1}, s_{i2}, s_{i3} are the components of the 3-dimensional vectors $\mathbf{s}_0$ and $\mathbf{s}_i$ respectively. If the nucleons have non-relativistic velocities we can, as is well known, put

$$\varrho_3 = -1 \quad \text{and} \quad \varrho_1 = \varrho_2 = 0. \qquad (336)$$

With the notations in eqs. (332) -(335) we have from eqs. (64)—(71) and (331):

$$\begin{aligned}\pi_\beta = & - i\,\bar{c}_1'\,s_3\,\bar{\omega}_5 + \\ & + i\,\bar{c}_1''\,s_3\,\bar{\omega}_4 - \\ & - i\,\bar{c}_2'\,s_0\,\bar{\beta}_4 - \\ & - \bar{c}_2' \sum_{k=1}^{3} s_{1k}\,\bar{\beta}_k + \\ & + \bar{c}_2''\,s_0\,\bar{\beta}_4\,(\bar{\omega}_4 - \bar{\omega}_5) - \\ & - i\,\bar{c}_2'' \sum_{k=1}^{3} s_{1k}\,\bar{\beta}_k\,(\bar{\omega}_4 - \bar{\omega}_5) - \\ & - \bar{c}_3 \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{3kl}\,\beta_k\,\bar{\beta}_l - \\ & - i\,\bar{c}_3 \sum_{k=1}^{3} s_{2k}\,(\bar{\beta}_4\,\bar{\beta}_k - \bar{\beta}_k\,\bar{\beta}_4).\end{aligned} \tag{337}$$

Inserting this expression in equ. (327) we get:

$$\begin{aligned}\Psi_n = & - \frac{1}{\bar{\mu}\,c}\,p_n\,\Psi_5 - \\ & - \frac{i}{\bar{\mu}\,c}\Big[- i\,\bar{c}_1'\,s_3\,(\bar{\omega}_5\,\Psi)_n + \\ & + i\,\bar{c}_1''\,s_3\,(\bar{\omega}_4\,\Psi)_n + \\ & + \bar{c}_2'\,s_0 \sum_{n'=1}^{5} (\delta_{n4}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'4})\,\Psi_{n'} - \\ & - i\,\bar{c}_2' \sum_{k=1}^{3} s_{1k} \sum_{n'=1}^{5} (\delta_{nk}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'k})\,\Psi_{n'} + \\ & + i\,\bar{c}_2''\,s_0 \sum_{n'=1}^{5} (\delta_{n4}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'4})\,((\bar{\omega}_4 - \bar{\omega}_5)\,\Psi)_{n'} + \\ & + \bar{c}_2'' \sum_{k=1}^{3} s_{1k} \sum_{n'=1}^{5} (\delta_{nk}\,\delta_{n'5} - \delta_{n5}\,\delta_{n'k})\,((\bar{\omega}_4 - \bar{\omega}_5)\,\Psi)_{n'} - \\ & - \bar{c}_3 \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{3kl} \sum_{n'=1}^{5} \delta_{nk}\,\delta_{n'l}\,\Psi_{n'} - \\ & - i\,\bar{c}_3 \sum_{k=1}^{3} s_{2k} \sum_{n'=1}^{5} (\delta_{n4}\,\delta_{n'k} - \delta_{nk}\,\delta_{n'4})\,\Psi_{n'}\Big].\end{aligned} \tag{338}$$

After summation we get since $n = 1, 2, 3$:

$$\begin{aligned} \Psi_n = \frac{1}{\bar{\mu} c} \Big[& - p_n \Psi_5 + \\ & + \bar{c}_1'' s_3 \Psi_n - \\ & - \bar{c}_2' s_{1n} \Psi_5 + \\ & + i \bar{c}_2'' s_{1n} \Psi_5 + \qquad (n = 1, 2, 3) \\ & + i \bar{c}_3 \sum_{k=1}^{3} {}^{(k \neq n)} s_{3nk} \Psi_k + \\ & + \bar{c}_3 s_{2n} \Psi_4 \Big] . \end{aligned} \tag{339}$$

This is a system of equations, which is linear in the redundant components Ψ_1, Ψ_2 and Ψ_3, and which can consequently be solved with respect to them.

The same is the case for the adjoint redundant components $\tilde{\Psi}_1$, $\tilde{\Psi}_2$ and $\tilde{\Psi}_3$, for which the corresponding equations are:

$$\begin{aligned} \tilde{\Psi}_n = \frac{1}{\bar{\mu} c} \Big[& \tilde{\Psi}_5 p_n + \\ & + \bar{c}_1'' s_3 \tilde{\Psi}_n + \\ & + \bar{c}_2' s_{1n} \tilde{\Psi}_5 + \qquad (n = 1, 2, 3) \\ & + i \bar{c}_2'' s_{1n} \tilde{\Psi}_5 - \\ & - i \bar{c}_3 \sum_{k=1}^{3} {}^{(k \neq n)} s_{3nk} \tilde{\Psi}_k - \\ & - \bar{c}_3 s_{2n} \tilde{\Psi}_4 \Big] . \end{aligned} \tag{340}$$

The expressions for Ψ_1, Ψ_2, Ψ_3, $\tilde{\Psi}_1$, $\tilde{\Psi}_2$, $\tilde{\Psi}_3$ given by eqs. (339) and (340) are now to be inserted in the last two terms of equ. (330), giving a Hamiltonian, which is a function only of the non-redundant components Ψ_4, Ψ_5, $\tilde{\Psi}_4$, $\tilde{\Psi}_5$ and their spatial derivatives.

As an example of this rigorous elimination we will here treat the simple case, when only $\bar{c}_1'$ and $\bar{c}_1''$ are different from zero and all the other $\bar{c}_k$:s vanish. Eqs. (339) and (340) are then:

$$\Psi_n = - \frac{1}{\bar{\mu} c} p_n \Psi_5 + \frac{\bar{c}_1'' s_3}{\bar{\mu} c} \Psi_n \qquad (n = 1, 2, 3), \tag{341}$$

and

$$\tilde{\Psi}_n = \frac{1}{\bar{\mu} c} \tilde{\Psi}_5 p_n + \frac{\bar{c}_1'' s_3}{\bar{\mu} c} \tilde{\Psi}_n \qquad (n = 1, 2, 3) \tag{342}$$

or

$$\Psi_n = -\frac{1}{\bar{\mu} c - \bar{c}_1'' s_3} p_n \Psi_5 \qquad (n = 1, 2, 3) \tag{343}$$

$$\tilde{\Psi}_n = \frac{1}{\bar{\mu} c - \bar{c}_1'' s_3} \tilde{\Psi}_5 p_n \qquad (n = 1, 2, 3) \tag{344}$$

If we insert these expressions in the last two terms of equ. (330) we get in addition to the Hamiltonian of the free space:

$$\int \left[-\bar{c}_1' c s_3 \tilde{\Psi}_5 \Psi_5 + \bar{c}_1'' c s_3 \tilde{\Psi}_4 \Psi_4 + \bar{c}_1'' c s_3 \sum_{k=1}^{3} \tilde{\Psi}_k \Psi_k - \frac{(\bar{c}_1'' s_3)^2}{\bar{\mu}} \sum_{k=1}^{3} \tilde{\Psi}_k \Psi_k \right] d V =$$

$$= \int \left[\bar{c}_1' c s_3 \Psi_5^* \Psi_5 - \bar{c}_1'' c s_3 \Psi_4^* \Psi_4 - \frac{\bar{c}_1'' s_3}{\bar{\mu} (\bar{\mu} c - \bar{c}_1'' s_3)} \sum_{k=1}^{3} \tilde{\Psi}_5 p_k^2 \Psi_5 \right] d V,$$

or finally:

$$\int \left[\bar{c}_1' c s_3 \Psi_5^* \Psi_5 - \bar{c}_1'' c s_3 \Psi_4^* \Psi_4 + \frac{\hbar^2}{\bar{\mu}} \frac{\bar{c}_1'' s_3}{\bar{\mu} c - \bar{c}_1'' s_3} \sum_{k=1}^{3} \frac{\partial \Psi_5^*}{\partial x_k} \cdot \frac{\partial \Psi_5}{\partial x_k} \right] d V, \tag{345}$$

where the $\bar{c}_1'$-term is the interaction term treated by WENTZEL [32].

In the weak coupling case, when the additional term to the Hamiltonian is a small perturbation, i. e. when the $\bar{c}_k$:s are small, the additional term (345) is:

$$\int s_3 \left[\bar{c}_1' c \Psi_5^* \Psi_5 + \bar{c}_1'' \frac{\hbar^2}{\bar{\mu}^2 c} \sum_{k=1}^{3} \frac{\partial \Psi_5^*}{\partial x_k} \frac{\partial \Psi_5}{\partial x_k} - \bar{c}_1'' c \Psi_4^* \Psi_4 \right] d V. \tag{346}$$

In the other limit of approximation, the strong coupling case, when the interaction constants are very big, the $\bar{c}_1''$-term can give two results. On the one hand we can make $\bar{c}_1''$ very big, in which case the $\Psi_4^* \Psi_4$-term dominates over the $\sum_{k=1}^{3} \frac{\partial \Psi_5^*}{\partial x_k} \frac{\partial \Psi_5}{\partial x_k}$-term, since the factor before the latter term tends to the limited constant $-\frac{\hbar^2}{\bar{\mu}}$. On the other hand we can make $\bar{c}_1'' s_3$ very nearly equal to $\bar{\mu} c$ in the region where s_3 is important. In this case we can make the $\sum_{k=1}^{3} \frac{\partial \Psi_5^*}{\partial x_k} \cdot \frac{\partial \Psi_5}{\partial x_k}$-term dominate as much as we want over the $\Psi_4^* \Psi_4$-term. This latter case has been treated by PAULI & HU [27].

We will now eliminate the redundant components for all the terms in (337) in the weak coupling case, when the coefficients $\bar{c}_k$ are so small that it is sufficient to retain only those terms in (330) which are linear in the $\bar{c}_k$:s. which of course means that we neglect contact interactions.

It is immediately clear that the last term of (330) can only give terms which are at least quadratic in the $\bar{c}_k$:s, and we can consequently confine our investigation to the term

$$-i c \int \tilde{\Psi} \bar{\pi}_\beta \Psi d V. \tag{347}$$

The π in this expression always gives rise to a $\bar{c}_k$-factor, and as we are only interested in terms which are linear in the $\bar{c}_k$:s, we can, by eliminating the redundant components, use only the $\bar{c}_k$-independent terms in the expressions for Ψ_1, Ψ_2 and Ψ_3. If we write the solutions of eqs. (339) and (340) as a power-series in the $\bar{c}_k$:s, it is evident that the $\bar{c}_k$-independent terms are:

$$\Psi_n = -\frac{1}{\bar{\mu} c} p_n \Psi_5 = \frac{i}{\bar{\varkappa}} \frac{\partial \Psi_5}{\partial x_n} \qquad (n = 1, 2, 3) \tag{348}$$

$$\tilde{\Psi}_n = \frac{1}{\bar{\mu} c} \tilde{\Psi}_5 p_n = \frac{i}{\bar{\varkappa}} \frac{\partial \tilde{\Psi}_5}{\partial x_n} \qquad (n = 1, 2, 3) \tag{349}$$

which approximation is to be used in (347).

We have therefore in the indicated approximation:

$$\begin{aligned}
& -ic \int \tilde{\Psi} \bar{\pi}_\beta \Psi \, dV = \\
& = -ic \int \Big[-i \bar{c}_1' s_3 \tilde{\Psi}_5 \Psi_5 + \\
& + i \bar{c}_1'' s_3 \sum_{n=1}^{3} \tilde{\Psi}_n \Psi_n + i \bar{c}_1'' s_3 \tilde{\Psi}_4 \Psi_4 + \\
& + \bar{c}_2' s_0 \sum_{n=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_n (\delta_{n4} \delta_{n'5} - \delta_{n5} \delta_{n'4}) \Psi_{n'} - \\
& - i \bar{c}_2' \sum_{k=1}^{3} s_{1k} \sum_{n=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_n (\delta_{nk} \delta_{n'5} - \delta_{n5} \delta_{n'k}) \Psi_{n'} + \\
& + i \bar{c}_2'' s_0 \sum_{n=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_n (\delta_{n4} \delta_{n'5} - \delta_{n5} \delta_{n'4}) ((\bar{\omega}_4 - \bar{\omega}_5) \Psi)_{n'} + \\
& + \bar{c}_2'' \sum_{k=1}^{3} s_{1k} \sum_{n=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_n (\delta_{nk} \delta_{n'5} - \delta_{n5} \delta_{n'k}) ((\bar{\omega}_4 - \bar{\omega}_5) \Psi)_{n'} - \\
& - \bar{c}_3 \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{3kl} \sum_{n=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_n \delta_{nk} \delta_{n'l} \Psi_{n'} - \\
& - i \bar{c}_3 \sum_{k=1}^{3} s_{2k} \sum_{n=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_n (\delta_{n4} \delta_{n'k} - \delta_{nk} \delta_{n'4}) \Psi_{n'} \Big] dV = \\
& = \int \Big[\bar{c}_1' s_3 c \Psi_5^* \Psi_5 + \\
& + \bar{c}_1'' s_3 \cdot \frac{c}{\bar{\varkappa}^2} \cdot \sum_{k=1}^{3} \frac{\partial \Psi_5^*}{\partial x_k} \cdot \frac{\partial \Psi_5}{\partial x_k} - \bar{c}_1'' s_3 c \Psi_4^* \Psi_4 +
\end{aligned}$$

$$+ i \bar{c}_2' s_0 c (\Psi_4^* \Psi_5 - \Psi_5^* \Psi_4) + \tag{350}$$

$$+ i \bar{c}_2' \sum_{k=1}^{3} s_{1k} \frac{c}{\varkappa} \left(\frac{\partial \Psi_5^*}{\partial x_k} \Psi_5 - \Psi_5^* \frac{\partial \Psi_5}{\partial x_k} \right) +$$

$$+ \bar{c}_2'' s_0 c (\Psi_5^* \Psi_4 + \Psi_4^* \Psi_5) +$$

$$+ \bar{c}_2'' \sum_{k=1}^{3} s_{1k} \frac{c}{\varkappa} \left(\frac{\partial \Psi_5^*}{\partial x_k} \Psi_5 + \Psi_5^* \frac{\partial \Psi_5}{\partial x_k} \right) +$$

$$+ i \bar{c}_3 \tfrac{1}{2} \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{3kl} \frac{c}{\varkappa^2} \left(\frac{\partial \Psi_5^*}{\partial x_k} \frac{\partial \Psi_5}{\partial x_l} - \frac{\partial \Psi_5^*}{\partial x_l} \frac{\partial \Psi_5}{\partial x_k} \right) +$$

$$+ i \bar{c}_3 \sum_{k=1}^{3} s_{2k} \frac{c}{\varkappa} \left(\Psi_4^* \frac{\partial \Psi_5}{\partial x_k} - \frac{\partial \Psi_5^*}{\partial x_k} \Psi_4 \right) \Bigg] d V.$$

We will now introduce the new variables U and P defined by the equations

$$\Psi_5 = \frac{1}{\hbar} \sqrt{\frac{\bar{\mu}}{8\pi}} \cdot U, \tag{351}$$

$$\Psi_4 = \sqrt{\frac{8\pi}{\bar{\mu}}} \cdot P^*. \tag{352}$$

Since from equ. (278)

$$\Pi_4 = \sum_{n=1}^{5} i\, \delta_{44}\, \delta_{n5}\, \Psi_n = i\, \Psi_5 \tag{353}$$

equ. (280) gives:

$$[\Psi_5 (\boldsymbol{r}'),\ \Psi_4^* (\boldsymbol{r}'')] = i\, \delta (\boldsymbol{r}' - \boldsymbol{r}''). \tag{354}$$

This equation gives for U and P:

$$[U (\boldsymbol{r}'),\ P (\boldsymbol{r}'')] = i \hbar\, \delta (\boldsymbol{r}' - \boldsymbol{r}''), \tag{355}$$

which is the ordinary commutation rule for a variable U and its canonically conjugate momentum P.

If we introduce the dimensionless interaction constants $\bar{C}_k$, defined by the equations

$$\bar{c}_k = \bar{C}_k \frac{e^2}{\bar{\mu} c^2} \cdot \frac{8 \pi \hbar}{\varkappa}, \tag{356}$$

we finally obtain for the Hamiltonian:

$$H = \int \left[\frac{1}{8\pi} \operatorname{grad} U^* \ \operatorname{grad}\ U + 8 \pi c^2 P P^* + \frac{\varkappa^2}{8\pi}\ U^* U \right] d V +$$

$$+ \int \frac{e^2}{\bar{\mu} c^2} \Big[\bar{C}_1' s_3 \{U^* U\} +$$

$$
\begin{aligned}
&+ \bar{C}_1'' s_3 \frac{1}{\bar{\varkappa}^2} \{\mathrm{grad}\, U^* \ \mathrm{grad}\, U - (8\pi c)^2 P P^*\} + \\
&+ i \bar{C}_2' s_0 \frac{8\pi c}{\bar{\varkappa}} \{P U - U^* P^*\} + \\
&+ i \bar{C}_2' \mathbf{s}_1 \frac{1}{\bar{\varkappa}} \{\mathrm{grad}\, U^* U - U^* \ \mathrm{grad}\, U\} + \qquad (357) \\
&+ \bar{C}_2'' s_0 \frac{8\pi c}{\bar{\varkappa}} \{P U + U^* P^*\} + \\
&+ \bar{C}_2'' \mathbf{s}_1 \frac{1}{\bar{\varkappa}} \{\mathrm{grad}\, U^* U + U^* \ \mathrm{grad}\, U\} + \\
&+ i \bar{C}_3 \mathbf{s}_3 \frac{1}{\bar{\varkappa}^2} \{\mathrm{grad}\, U^* \times \mathrm{grad}\, U\} + \\
&+ i \bar{C}_3 \mathbf{s}_2 \frac{8\pi c}{\bar{\varkappa}^2} \{P \ \mathrm{grad}\, U - \mathrm{grad}\, U^* P^*\} \Big] \, d V.
\end{aligned}
$$

As was mentioned above the $\bar{C}_1'$-term has been treated by WENTZEL [32] and the $\bar{C}_1''$-term by PAULI & HU [27]. Moreover the $\bar{C}_3$-term has been treated in the weak coupling approximation by JAUCH & LOPES [16].

CHAPTER VIII

Elimination of the redundant components in the vector pair theory

If we assume $\mu_4 = \bar{\bar{\mu}}_5 = \mu$, the equations of definition (180), (181), (189) and (190) for the redundant components in the vector theory are:

$$\Psi_{kn} = - \frac{1}{\bar{\bar{\mu}} c} (p_k \Psi_{5n} - p_n \Psi_{5k}) - \frac{i}{\bar{\bar{\mu}} c} (\bar{\bar{\pi}}_\beta \Psi)_{kn} \quad (k, n = 1, 2, 3), \quad (358)$$

$$\Psi_{54} = - \frac{1}{\bar{\bar{\mu}} c} \sum_{k=1}^{3} p_k \Psi_{4k} - \frac{i}{\bar{\bar{\mu}} c} (\bar{\bar{\pi}}_\beta \Psi)_{54}, \quad (359)$$

$$\tilde{\Psi}_{kn} = \frac{1}{\bar{\bar{\mu}} c} (\tilde{\Psi}_{5n} p_k - \tilde{\Psi}_{5k} p_n) - \frac{i}{\bar{\bar{\mu}} c} (\tilde{\Psi} \bar{\bar{\pi}}_\beta)_{kn} \quad (k, n = 1, 2, 3), \quad (360)$$

$$\tilde{\Psi}_{54} = \frac{1}{\bar{\bar{\mu}} c} \sum_{k=1}^{3} \tilde{\Psi}_{4k} p_k - \frac{i}{\bar{\bar{\mu}} c} (\tilde{\Psi} \bar{\bar{\pi}}_\beta)_{54}. \quad (361)$$

The Hamiltonian is:

$$H = ic \int \tilde{\Psi} \left(\sum_{k=1}^{3} \bar{\bar{\beta}}_k p_k + i \bar{\bar{\mu}} c - \bar{\bar{\pi}}_\beta \right) \Psi \, dV =$$

$$= ic \int \Bigg[\sum_{k=1}^{3} \sum_{m=4}^{5} \sum_{n=1}^{3} \sum_{m'=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_{mn} \, i \left(- \delta_{m5} \delta_{m'k} \delta_{nn'} - \delta_{nk} \delta_{m'5} \delta_{mn'} \right) p_k \Psi_{m'n'} +$$

$$+ i \mu c \sum_{m=4}^{5} \sum_{n=1}^{3} \tilde{\Psi}_{mn} \Psi_{mn} - \sum_{m=4}^{5} \sum_{n=1}^{3} \tilde{\Psi}_{mn} (\pi_\beta \Psi)_{mn} \Bigg] dV =$$

$$= \int \Bigg[c \sum_{k=1}^{3} \sum_{n=1}^{3} \tilde{\Psi}_{5n} p_k \Psi_{kn} + c \sum_{k=1}^{3} \tilde{\Psi}_{4k} p_k \Psi_{54} - \bar{\bar{\mu}} c^2 \sum_{m=4}^{5} \sum_{n=1}^{3} \tilde{\Psi}_{mn} \Psi_{mn} -$$

$$- ic \sum_{m=4}^{5} \sum_{n=1}^{3} \tilde{\Psi}_{mn} (\bar{\bar{\pi}}_\beta \Psi)_{mn} \Bigg] dV =$$

$$= \int \Bigg[- \frac{1}{\bar{\bar{\mu}}} \sum_{k=1}^{3} \sum_{n=1}^{3} \tilde{\Psi}_{5n} p_k (p_k \Psi_{5n} - p_n \Psi_{5k}) -$$

$$- \frac{i}{\bar{\bar{\mu}}} \sum_{k=1}^{3} \sum_{n=1}^{3} \tilde{\Psi}_{5n} p_k (\bar{\bar{\pi}}_\beta \Psi)_{kn} - \frac{1}{\bar{\bar{\mu}}} \sum_{k=1}^{3} \sum_{n=1}^{3} \tilde{\Psi}_{4k} p_k p_n \Psi_{4n} -$$

$$- \frac{i}{\bar{\bar{\mu}}} \sum_{k=1}^{3} \tilde{\Psi}_{4k} p_k (\bar{\bar{\pi}}_\beta \Psi)_{54} - \bar{\mu} c^2 \sum_{k=1}^{3} \tilde{\Psi}_{4k} \Psi_{4k} - \bar{\bar{\mu}} c^2 \sum_{k=1}^{3} \tilde{\Psi}_{5k} \Psi_{5k} -$$

$$- ic \sum_{m=4}^{5} \sum_{n=1}^{3} \tilde{\Psi}_{mn} (\bar{\bar{\pi}}_\beta \Psi)_{mn} \Bigg] dV,$$

and finally:

$$H = \int \Bigg[\frac{\hbar^2}{2 \bar{\bar{\mu}}} \sum_{k=1}^{3} \sum_{n=1}^{3} \left(\frac{\partial \Psi_{5n}^*}{\partial x_k} - \frac{\partial \Psi_{5k}^*}{\partial x_n} \right) \left(\frac{\partial \Psi_{5n}}{\partial x_k} - \frac{\partial \Psi_{5k}}{\partial x_n} \right) +$$

$$+ \frac{\hbar^2}{\bar{\bar{\mu}}} \sum_{k=1}^{3} \sum_{n=1}^{3} \frac{\partial \Psi_{4k}^*}{\partial x_k} \frac{\partial \Psi_{4n}}{\partial x_n} + \bar{\bar{\mu}} c^2 \sum_{k=1}^{3} \Psi_{4k}^* \Psi_{4k} + \mu c^2 \sum_{k=1}^{3} \Psi_{5k}^* \Psi_{5k} - \tag{362}$$

$$- ic \tilde{\Psi} \bar{\pi}_\beta \Psi + \frac{1}{\mu} (\tilde{\Psi} \bar{\pi}_\beta)_{54} (\bar{\pi}_\beta \Psi)_{54} + \frac{1}{2 \mu} \sum_{k=1}^{3} \sum_{n=1}^{3} (\tilde{\Psi} \bar{\bar{\pi}}_\beta)_{kn} (\bar{\bar{\pi}}_\beta \Psi)_{kn} \Bigg] dV.$$

Putting $\pi_\beta = 0$ in this expression we get the positive Hamiltonian of the vector meson field in free space, and the quantization in accordance with Chapter VI is thus possible, if we assume the terms containing π_β in equ. (362) to be small in comparison with the other terms.

The most general form of $\bar{\bar{\pi}}_\beta$ is according to eqs. (246)—(254):

$$\begin{aligned}
\bar{\bar{\pi}}_\beta = {} & i\bar{\bar{c}}_1' \, \tilde{\psi}\, \psi\, \bar{\bar{\omega}}_5 + \\
& + i\, \bar{\bar{c}}_1'' \, \tilde{\psi}\, \psi\, \bar{\bar{\omega}}_4 - \\
& - i\, c_2' \sum_{k=1}^{4} \tilde{\psi}\, \gamma_k\, \psi\, \beta_k + \\
& + \bar{\bar{c}}_2'' \sum_{k=1}^{4} \tilde{\psi}\, \gamma_k\, \psi\, \bar{\bar{\beta}}_k\, (\bar{\bar{\omega}}_4 - \bar{\bar{\omega}}_5) + \\
& + i\, c_3' \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \tilde{\psi}\, \gamma_k\, \gamma_l\, \psi\, \bar{\bar{\beta}}_k\, \bar{\beta}_l\, \bar{\bar{\omega}}_4 + \\
& + i\, c_3'' \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \tilde{\psi}\, \gamma_k\, \gamma_l\, \psi\, \bar{\bar{\beta}}_k\, \beta_l\, \omega_5 + \\
& + i\, c_4' \cdot \tfrac{1}{2} \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \tilde{\psi}\, \gamma_k\, \gamma_l\, \gamma_m\, \psi\, \bar{\bar{\beta}}_k\, \bar{\beta}_l\, \bar{\bar{\beta}}_m + \\
& + \bar{c}_4'' \cdot \tfrac{1}{2} \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \tilde{\psi}\, \gamma_k\, \gamma_l\, \gamma_m\, \psi\, \bar{\bar{\beta}}_k\, \bar{\beta}_l\, \bar{\beta}_m\, (\bar{\bar{\omega}}_4 - \bar{\bar{\omega}}_5) + \\
& + i\, c_5\, \tfrac{1}{4} \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \tilde{\psi}\, \gamma_k\, \gamma_l\, \gamma_m\, \gamma_n\, \psi\, \beta_k\, \beta_l\, \beta_m\, \bar{\beta}_n\, .
\end{aligned} \tag{363}$$

If we now introduce the s_i:s of eqs. (332)—(335) the c_1-, c_2- and c_3-terms give a result analogous to that obtained for scalar mesons in equ. (337).

The c_4- and $\bar{\bar{c}}_5$-terms are more complicated. We have:

$$\begin{aligned}
& \tfrac{1}{2} \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \tilde{\psi}\, \gamma_k\, \gamma_l\, \gamma_m\, \psi\, \bar{\bar{\beta}}_k\, \bar{\bar{\beta}}_l\, \bar{\bar{\beta}}_m = \\
& = \tfrac{1}{2} \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' i\, s_{0kl}\, (\bar{\bar{\beta}}_4\, \bar{\bar{\beta}}_k\, \bar{\bar{\beta}}_l - \bar{\bar{\beta}}_k\, \bar{\bar{\beta}}_4\, \bar{\bar{\beta}}_l + \bar{\bar{\beta}}_k\, \bar{\bar{\beta}}_l\, \bar{\bar{\beta}}_4) + \\
& + \tfrac{1}{2}\, s_1\, (\beta_1\, \beta_2\, \bar{\bar{\beta}}_3 + \text{even perm.} - \text{odd perm.}) = \\
& = i \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{0kl}\, (\bar{\bar{\beta}}_4\, \bar{\bar{\beta}}_k\, \bar{\bar{\beta}}_l - \tfrac{1}{2}\, \bar{\bar{\beta}}_k\, \bar{\beta}_4\, \bar{\bar{\beta}}_l) + \\
& + \tfrac{1}{2}\, s_1\, (\bar{\beta}_1\, \bar{\bar{\beta}}_2\, \bar{\bar{\beta}}_3 - \bar{\bar{\beta}}_3\, \bar{\bar{\beta}}_2\, \bar{\beta}_1 + \text{even perm.}) = \\
& = i \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{0kl}\, (\bar{\bar{\beta}}_4\, \bar{\bar{\beta}}_k\, \bar{\bar{\beta}}_l - \tfrac{1}{2}\, \bar{\bar{\beta}}_k\, \bar{\bar{\beta}}_4\, \bar{\bar{\beta}}_l) + \\
& + s_1\, (\bar{\bar{\beta}}_1\, \bar{\bar{\beta}}_2\, \bar{\bar{\beta}}_3 + \text{cycl. perm.}),
\end{aligned}$$

since $\beta_1\beta_2\beta_3 = -\beta_3\bar{\beta}_2\bar{\bar{\beta}}_1$ and since the cyclic permutations give all the even permutations.

Further we have:

$$\tfrac{1}{4}\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\sum_{m=1}^{4}{}'\sum_{n=1}^{4}{}'\tilde{\psi}\gamma_k\gamma_l\gamma_m\gamma_n\psi\beta_k\bar{\beta}_l\bar{\bar{\beta}}_m\bar{\bar{\beta}}_n =$$

$$= -\frac{i}{4}s_2(\bar{\bar{\beta}}_1\bar{\beta}_2\bar{\bar{\beta}}_3\bar{\bar{\beta}}_4 + \text{even perm.} - \text{odd perm.}) =$$

$$= \frac{i}{4}s_2(\bar{\bar{\beta}}_4\bar{\bar{\beta}}_1\bar{\bar{\beta}}_2\bar{\bar{\beta}}_3 - \bar{\bar{\beta}}_1\bar{\bar{\beta}}_4\bar{\bar{\beta}}_2\bar{\bar{\beta}}_3 - \bar{\bar{\beta}}_2\bar{\bar{\beta}}_1\bar{\bar{\beta}}_4\beta_3 +$$

$$+ \bar{\bar{\beta}}_1\bar{\bar{\beta}}_3\bar{\bar{\beta}}_2\bar{\bar{\beta}}_4 + \text{even perm. of } 1,\ 2,\ 3 - \text{odd perm. of } 1,\ 2,\ 3) =$$

$$= \frac{i}{2}s_2(\beta_4\beta_1\beta_2\beta_3 - \beta_1\bar{\beta}_4\beta_2\beta_3 + \text{even perm. of } 1,2,3 - \text{odd perm. of } 1,2,3) =$$

$$= \frac{i}{2}s_2(\bar{\bar{\beta}}_4\bar{\bar{\beta}}_1\bar{\bar{\beta}}_2\bar{\beta}_3 - \bar{\bar{\beta}}_4\bar{\bar{\beta}}_3\bar{\bar{\beta}}_2\bar{\beta}_1 - \bar{\bar{\beta}}_1\bar{\bar{\beta}}_4\bar{\bar{\beta}}_2\bar{\bar{\beta}}_3 + \bar{\bar{\beta}}_2\bar{\bar{\beta}}_4\bar{\bar{\beta}}_1\bar{\bar{\beta}}_3 +$$

$$+ \text{even perm. of } 1,\ 2,\ 3) =$$

$$= i\,s_2(\bar{\bar{\beta}}_4\bar{\bar{\beta}}_1\bar{\bar{\beta}}_2\bar{\bar{\beta}}_3 - \bar{\bar{\beta}}_1\bar{\bar{\beta}}_4\bar{\bar{\beta}}_2\bar{\bar{\beta}}_3 + \text{cycl. perm. of } 1,\ 2,\ 3).$$

Using these results we can write the expression (363) for the most general $\bar{\pi}_\beta$ in the following way:

$$\begin{aligned}
\bar{\pi}_\beta = &- i\,\bar{\bar{c}}_1'\,s_3\,\bar{\omega}_5 - \\
&- i\,\bar{c}_1''\,s_3\,\omega_4 - \\
&- i\,c_2'\,s_0\,\bar{\beta}_4 - \\
&- \bar{c}_2'\sum_{k=1}^{3}s_{1k}\bar{\bar{\beta}}_k + \\
&+ \bar{\bar{c}}_2''\,s_0\,\bar{\bar{\beta}}_4(\bar{\bar{\omega}}_4 - \bar{\bar{\omega}}_5) - \\
&- i\,\bar{\bar{c}}_2''\sum_{k=1}^{3}s_{1k}\bar{\bar{\beta}}_k(\bar{\bar{\omega}}_4 - \bar{\bar{\omega}}_5) + \\
&+ \bar{c}_3'\sum_{k=1}^{3}{}'\sum_{l=1}^{3}{}'s_{3kl}\beta_k\beta_l\omega_4 + \\
&+ i\,\bar{\bar{c}}_3'\sum_{k=1}^{3}s_{2k}(\bar{\bar{\beta}}_4\bar{\beta}_k - \bar{\bar{\beta}}_k\bar{\bar{\beta}}_4)\bar{\bar{\omega}}_4 + \qquad (364)\\
&+ \bar{c}_3''\sum_{k=1}^{3}{}'\sum_{l=1}^{3}{}'s_{3kl}\bar{\beta}_k\bar{\beta}_l\bar{\bar{\omega}}_5 +
\end{aligned}$$

$$+ i \bar{\bar{c}}_3'' \sum_{k=1}^{3} s_{2k} (\bar{\bar{\beta}}_4 \bar{\bar{\beta}}_k - \bar{\bar{\beta}}_k \bar{\bar{\beta}}_4) \bar{\bar{\omega}}_5 -$$

$$- \bar{\bar{c}}_4' \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{0kl} (\bar{\bar{\beta}}_4 \bar{\bar{\beta}}_k \bar{\bar{\beta}}_l - \tfrac{1}{2} \bar{\bar{\beta}}_k \bar{\bar{\beta}}_4 \bar{\bar{\beta}}_l) +$$

$$+ i \bar{\bar{c}}_4' s_1 (\beta_1 \bar{\bar{\beta}}_2 \bar{\bar{\beta}}_3 + \text{cycl. perm.}) +$$

$$+ i \bar{\bar{c}}_4'' \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{0kl} (\bar{\bar{\beta}}_4 \bar{\bar{\beta}}_k \bar{\bar{\beta}}_l - \tfrac{1}{2} \bar{\bar{\beta}}_k \bar{\bar{\beta}}_4 \bar{\beta}_l) (\bar{\bar{\omega}}_4 - \bar{\bar{\omega}}_5) +$$

$$+ \bar{\bar{c}}_4'' s_1 \cdot (\bar{\bar{\beta}}_1 \bar{\bar{\beta}}_2 \bar{\bar{\beta}}_3 + \text{cycl. perm.}) (\bar{\bar{\omega}}_4 - \bar{\bar{\omega}}_5) -$$

$$- \bar{\bar{c}}_5 s_2 (\bar{\bar{\beta}}_4 \beta_1 \bar{\bar{\beta}}_2 \bar{\bar{\beta}}_3 - \bar{\bar{\beta}}_1 \bar{\bar{\beta}}_4 \bar{\bar{\beta}}_2 \bar{\bar{\beta}}_3 + \text{cycl. perm. of } 1, 2, 3).$$

Inserting this expression for π_β in eqs. (358)—(361), we obtain eight equations, which are linear with respect to the redundant components Ψ_{23}, Ψ_{31}, Ψ_{12}, Ψ_{54}, $\tilde{\Psi}_{23}$, $\tilde{\Psi}_{31}$, $\tilde{\Psi}_{12}$, $\tilde{\Psi}_{54}$, and which we can solve in principle with respect to these components, although the result will in general be very complicated.

This exact elimination we will perform as an example only for the simple case $\bar{c}_1' \neq 0$ and all other $\bar{c}_k$:s vanishing. This assumption gives:

$$\Psi_{kn} = -\frac{1}{\bar{\bar{\mu}} c} (p_k \Psi_{5n} - p_n \Psi_{5k}) \qquad (n, k = 1, 2, 3), \tag{365}$$

$$\Psi_{54} = -\frac{1}{\bar{\bar{\mu}} c} \sum_{k=1}^{3} p_k \Psi_{4k} - \frac{\bar{\bar{c}}_1' s_3}{\bar{\bar{\mu}} c} \Psi_{54}, \tag{366}$$

$$\tilde{\Psi}_{kn} = \frac{1}{\bar{\bar{\mu}} c} (\tilde{\Psi}_{5n} p_k - \tilde{\Psi}_{5k} p_n) \qquad (n, k = 1, 2, 3), \tag{367}$$

$$\tilde{\Psi}_{54} = \frac{1}{\bar{\mu} c} \sum_{k=1}^{3} \tilde{\Psi}_{4k} p_k - \frac{c_1' s_3}{\bar{\bar{\mu}} c} \tilde{\Psi}_{54}, \tag{368}$$

or for the 54-components:

$$\Psi_{54} = -\frac{1}{\bar{\bar{\mu}} c + \bar{\bar{c}}_1' s_3} \sum_{k=1}^{3} p_k \Psi_{4k}, \tag{369}$$

$$\tilde{\Psi}_{54} = \frac{1}{\bar{\mu} c + \bar{\bar{c}}_1' s_3} \sum_{k=1}^{3} \tilde{\Psi}_{4k} p_k. \tag{370}$$

If we insert these expressions in the last three terms of equ. (362) we get for the additional term to the Hamiltonian of the free space:

$$\int \left[- c \bar{c}_1' s_3 \sum_{k=1}^{3} \tilde{\Psi}_{5k} \Psi_{5k} - \frac{\bar{c}_1' s_3}{\mu} (\bar{\mu} c + \bar{c}_1' s_3) \tilde{\Psi}_{54} \Psi_{54} \right] d V =$$

$$= \int \left[c \bar{\bar{c}}_1' s_3 \cdot \sum_{k=1}^{3} \Psi_{5k}^* \Psi_{5k} - \frac{\hbar^2}{\bar{\bar{\mu}}} \frac{\bar{\bar{c}}_1' s_3}{\bar{\bar{\mu}} c + \bar{\bar{c}}_1' s_3} \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial \Psi_{4k}^*}{\partial x_k} \frac{\partial \Psi_{4l}}{\partial x_l} \right] d V. \tag{371}$$

In the strong coupling approximation we can either make c_1' large or make $\bar{\bar{c}}_1'$ nearly equal to $-\frac{\bar{\bar{\mu}} c}{s_3}$ in the region where s_3 is important. In the first case, which has been treated by PAULI & HU [27], the $\sum\limits_{k=1}^{3} \Psi_{5k}^{*} \Psi_{5k}$-term dominates over the second term, and in the second case it is small compared with the second term.

In the weak coupling approximation, considered by BRULIN and the author [5], the corresponding additional term to the Hamiltonian is:

$$\int \bar{\bar{c}}_1' \, s_3 \, c \left[\sum_{k=1}^{3} \Psi_{5k}^{*} \Psi_{5k} - \frac{1}{\bar{\bar{\varkappa}}^2} \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial \Psi_{4k}^{*}}{\partial x_k} \cdot \frac{\partial \Psi_{4l}}{\partial x_l} \right] d V. \tag{372}$$

When eliminating the redundant components in the weak coupling case for all the terms in (364) we can, just as in the scalar case, confine ourselves to the term

$$- i c \int \tilde{\Psi} \bar{\bar{\pi}}_\beta \Psi \, d V \tag{373}$$

in the expression (362) for the Hamiltonian, since the other two terms containing $\bar{\bar{\pi}}_\beta$ are at least quadratic in the c_k:s. Again, this means of course that we neglect all contact interactions.

Expanding the solution of eqs. (358)—(361) in a power series in the $\bar{\bar{c}}_k$:s and retaining only the terms independent of $\bar{c}_k$, we obtain:

$$\Psi_{kn} = - \frac{1}{\bar{\bar{\mu}} c} (p_k \Psi_{5n} - p_n \Psi_{5k}) = \frac{i}{\bar{\bar{\varkappa}}} \left(\frac{\partial \Psi_{5n}}{\partial x_k} - \frac{\partial \Psi_{5k}}{\partial x_n} \right), \tag{374}$$

$$\Psi_{54} = - \frac{1}{\bar{\bar{\mu}} c} \sum_{k=1}^{3} p_k \Psi_{4k} = \frac{i}{\bar{\bar{\varkappa}}} \sum_{k=1}^{3} \frac{\partial \Psi_{4k}}{\partial x_k}, \tag{375}$$

$$\tilde{\Psi}_{kn} = \frac{1}{\mu c} (\tilde{\Psi}_{5n} p_k \quad \tilde{\Psi}_{5k} p_n) = \frac{i}{\varkappa} \left(\frac{\partial \tilde{\Psi}_{5n}}{\partial x_k} - \frac{\partial \tilde{\Psi}_{5k}}{\partial x_n} \right), \tag{376}$$

$$\tilde{\Psi}_{54} = \frac{1}{\bar{\bar{\mu}} c} \sum_{k=1}^{3} \tilde{\Psi}_{4k} p_k = \frac{i}{\bar{\bar{\varkappa}}} \sum_{k=1}^{3} \frac{\partial \tilde{\Psi}_{4k}}{\partial x_k}, \tag{377}$$

which approximation is to be inserted in the term (373), together with the expression (364) for $\bar{\bar{\pi}}_\beta$.

For the five last terms of (364) we have to calculate $\bar{\beta}_i \bar{\beta}_k \beta_l$ and $\bar{\bar{\beta}}_i \bar{\bar{\beta}}_k \beta_l \beta_r$. From the 4th member of equ. (145) we have, since $i \neq k \neq l \neq i$:

$$(mn \,|\, \bar{\bar{\beta}}_i \bar{\bar{\beta}}_k \bar{\bar{\beta}}_l \,|\, m'n') =$$
$$= i (\delta_{in'} \delta_{nk} \delta_{m5} \delta_{m'l} - \delta_{kn'} \delta_{nl} \delta_{mi} \delta_{m'5}) - \mathrm{ex}\,(mn) - \mathrm{ex}\,(m'n'),$$

which expression we shall use for the c_4 terms, at the same time exchanging the indices in a suitable way.

We have furthermore:

$$(mn\,|\,\beta_i\,\beta_k\,\beta_l\,\beta_r\,|\,m'n') =$$

$$= -\sum_{m''=1}^{5}\,\sum_{n''=1}^{5}(\delta_{in''}\,\delta_{nk}\,\delta_{m5}\,\delta_{m''l} - \delta_{kn''}\,\delta_{nl}\,\delta_{mi}\,\delta_{m''5})\,\cdot$$

$$\cdot\,(\delta_{m''r}\,\delta_{m'5}\,\delta_{n''n'} - \delta_{m''5}\,\delta_{m'r}\,\delta_{n''n'} - \delta_{n''r}\,\delta_{m'5}\,\delta_{m''n'} + \delta_{n''5}\,\delta_{m'r}\,\delta_{m''n'}) -$$

$$- \operatorname{ex}(mn) - \operatorname{ex}(m'n') =$$

$$= -\,\delta_{kn'}\,\delta_{nl}\,\delta_{mi}\,\delta_{m'r} - \operatorname{ex}(mn) - \operatorname{ex}(m'n').$$

Consequently, we obtain the following expression for the term (373):

$$-\,i\,c\int\tilde{\Psi}\,\bar{\pi}_\beta\,\Psi\,d\,V =$$

$$= -\,i\,c\int\Big[-\,i\,\bar{\bar{c}}_1'\,s_3\sum_{mn}\tilde{\Psi}_{mn}\,(\bar{\bar{\omega}}_5\,\Psi)_{mn} -$$

$$-\,i\,\bar{\bar{c}}_1''\,s_3\sum_{mn}\tilde{\Psi}_{mn}\,(\bar{\bar{\omega}}_4\,\Psi)_{mn} +$$

$$+\,c_2'\,s_0\sum_{m=1}^{5}\,\sum_{n=1}^{5}\,\sum_{m'=1}^{5}\,\sum_{n'=1}^{5}\tilde{\Psi}_{mn}\,(\delta_{m4}\,\delta_{m'5}\,\delta_{nn'} - \delta_{m5}\,\delta_{m'4}\,\delta_{nn'})\,\Psi_{m'n'} -$$

$$-\,i\,\bar{\bar{c}}_2'\sum_{k=1}^{3}s_{1k}\sum_{m=1}^{5}\,\sum_{n=1}^{5}\,\sum_{m'=1}^{5}\,\sum_{n'=1}^{5}\tilde{\Psi}_{mn}\,(\delta_{mk}\,\delta_{m'5}\,\delta_{nn'} - \delta_{m5}\,\delta_{m'k}\,\delta_{nn'})\,\Psi_{m'n'} +$$

$$+\,i\,\bar{\bar{c}}_2''\,s_0\sum_{m=1}^{5}\,\sum_{n=1}^{5}\,\sum_{m'=1}^{5}\,\sum_{n'=1}^{5}\tilde{\Psi}_{mn}\,(\delta_{m4}\,\delta_{m'5}\,\delta_{nn'} - \delta_{m5}\,\delta_{m'4}\,\delta_{nn'})\,((\bar{\bar{\omega}}_4 - \bar{\bar{\omega}}_5)\,\Psi)_{m'n'} +$$

$$+\,\bar{\bar{c}}_2''\sum_{k=1}^{3}s_{1k}\sum_{m=1}^{5}\,\sum_{n=1}^{5}\,\sum_{m'=1}^{5}\,\sum_{n'=1}^{5}\tilde{\Psi}_{mn}\,(\delta_{mk}\,\delta_{m'5}\,\delta_{nn'} - \delta_{m5}\,\delta_{m'k}\,\delta_{nn'})\,((\bar{\bar{\omega}}_4 - \bar{\bar{\omega}}_5)\,\Psi)_{m'n'} +$$

$$+\,c_3'\sum_{k=1}^{3}{}'\sum_{l=1}^{3}{}'\,s_{3kl}\sum_{m=1}^{5}\,\sum_{n=1}^{5}\,\sum_{m'=1}^{5}\,\sum_{n'=1}^{5}\tilde{\Psi}_{mn}\,\delta_{mk}\,\delta_{m'l}\,\delta_{nn'}\,(\omega_4\,\Psi)_{m'n'} +$$

$$+\,i\,\bar{\bar{c}}_3'\sum_{k=1}^{3}s_{2k}\sum_{m=1}^{5}\,\sum_{n=1}^{5}\,\sum_{m'=1}^{5}\,\sum_{n'=1}^{5}\tilde{\Psi}_{mn}\,(\delta_{m4}\,\delta_{m'k}\,\delta_{nn'} - \delta_{mk}\,\delta_{m'4}\,\delta_{nn'})\,(\bar{\omega}_4\,\Psi)_{m'n'} +$$

$$+\,\bar{c}_3''\sum_{k=1}^{3}{}'\sum_{l=1}^{3}{}'\,s_{3kl}\sum_{m=1}^{5}\,\sum_{n=1}^{5}\,\sum_{m'=1}^{5}\,\sum_{n'=1}^{5}\tilde{\Psi}_{mn}\,\delta_{mk}\,\delta_{m'l}\,\delta_{nn'}\,(\bar{\bar{\omega}}_5\,\Psi)_{m'n'} +$$

$$+\,i\,\bar{c}_3''\sum_{k=1}^{3}s_{2k}\sum_{m=1}^{5}\,\sum_{n=1}^{5}\,\sum_{m'=1}^{5}\,\sum_{n'=1}^{5}\tilde{\Psi}_{mn}\,(\delta_{m4}\,\delta_{m'k}\,\delta_{nn'} - \delta_{mk}\,\delta_{m'4}\,\delta_{nn'})\,(\omega_5\,\Psi)_{m'n'} -$$

$$-\,i\,c_4'\sum_{k=1}^{3}{}'\sum_{l=1}^{3}{}'\,s_{0kl}\sum_{m=1}^{5}\,\sum_{n=1}^{5}\,\sum_{m'=1}^{5}\,\sum_{n'=1}^{5}\tilde{\Psi}_{mn}\,(\delta_{m4}\,\delta_{nk}\,\delta_{m'5}\,\delta_{n'l} - \delta_{m5}\,\delta_{nk}\,\delta_{m'4}\,\delta_{n'l} +$$

$$+\,\tfrac{1}{2}\,\delta_{m5}\,\delta_{n4}\,\delta_{m'k}\,\delta_{n'l} + \tfrac{1}{2}\,\delta_{mk}\,\delta_{nl}\,\delta_{m'5}\,\delta_{n'4})\,\Psi_{m'n'} -$$

$$
\begin{aligned}
&- \bar{\bar{c}}{}_4' \, s_1 \sum_{m=1}^{5} \sum_{n=1}^{5} \sum_{m'=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_{mn} (\delta_{m5}\,\delta_{n2}\,\delta_{m'3}\,\delta_{n'1} + \delta_{m3}\,\delta_{n1}\,\delta_{m'5}\,\delta_{n'2} + \text{cycl. perm.})\, \Psi_{m'n'} - \\
&- c_4'' \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{0kl} \sum_{m=1}^{5} \sum_{n=1}^{5} \sum_{m'=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_{mn} (\delta_{m4}\,\delta_{nk}\,\delta_{m'5}\,\delta_{n'l} - \delta_{m5}\,\delta_{nk}\,\delta_{m'4}\,\delta_{n'l} + \\
&\qquad\qquad + \tfrac{1}{2}\,\delta_{m5}\,\delta_{n4}\,\delta_{m'k}\,\delta_{n'l} + \tfrac{1}{2}\,\delta_{mk}\,\delta_{nl}\,\delta_{m'5}\,\delta_{n'4})\, ((\bar{\omega}_4 - \bar{\omega}_5)\Psi)_{m'n'} + \\
&+ i\, c_4''\, s_1 \sum_{m=1}^{5} \sum_{n=1}^{5} \sum_{m'=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_{mn} (\delta_{m5}\,\delta_{n2}\,\delta_{m'3}\,\delta_{n'1} + \delta_{m3}\,\delta_{n1}\,\delta_{m'5}\,\delta_{n'2} + \\
&\qquad\qquad + \text{cycl. perm.})\, ((\bar{\omega}_4 - \bar{\bar{\omega}}_5)\Psi)_{m'n'} + \\
&+ \bar{c}_5\, s_2 \sum_{m=1}^{5} \sum_{n=1}^{5} \sum_{m'=1}^{5} \sum_{n'=1}^{5} \tilde{\Psi}_{mn} (\delta_{m4}\,\delta_{n2}\,\delta_{m'3}\,\delta_{n'1} + \delta_{m1}\,\delta_{n2}\,\delta_{m'4}\,\delta_{n'3} + \\
&\qquad\qquad + \text{cycl. perm. of } 1,\, 2,\, 3)\, \Psi_{m'n'} \Bigg]\, d\,V = \\
&= \int \Bigg[c_1'\, s_3\, c \left(\sum_{k=1}^{3} \Psi_{5k}^{*}\, \Psi_{5k} - \frac{1}{\varkappa^2} \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial \Psi_{4k}^{*}}{\partial x_k}\, \frac{\partial \Psi_{4l}}{\partial x_l} \right) + \\
&+ \bar{\bar{c}}{}_1''\, s_3\, c \left\{ \sum_{k=1}^{3} \Psi_{4k}^{*}\, \Psi_{4k} - \frac{1}{\varkappa^2}\, \frac{1}{2} \sum_{k=1}^{3} \sum_{l=1}^{3} \left(\frac{\partial \Psi_{5k}^{*}}{\partial x_l} - \frac{\partial \Psi_{5l}^{*}}{\partial x_k} \right) \left(\frac{\partial \Psi_{5k}}{\partial x_l} - \frac{\partial \Psi_{5l}}{\partial x_k} \right) \right\} + \\
&+ i\, \bar{\bar{c}}{}_2'\, s_0\, c \left(\sum_{k=1}^{3} \Psi_{4k}^{*}\, \Psi_{5k} - \sum_{k=1}^{3} \Psi_{5k}^{*}\, \Psi_{4k} \right) - \\
&- i\, \bar{\bar{c}}{}_2'\, c \sum_{k=1}^{3} s_{1k}\, \frac{1}{\varkappa} \left\{ \sum_{l=1}^{3} \left(\frac{\partial \Psi_{5k}^{*}}{\partial x_l} - \frac{\partial \Psi_{5l}^{*}}{\partial x_k} \right) \Psi_{5l} - \sum_{l=1}^{3} \Psi_{5l}^{*} \left(\frac{\partial \Psi_{5k}}{\partial x_l} - \frac{\partial \Psi_{5l}}{\partial x_k} \right) + \right. \\
&\qquad\qquad \left. + \Psi_{4k}^{*} \sum_{l=1}^{3} \frac{\partial \Psi_{4l}}{\partial x_l} - \sum_{l=1}^{3} \frac{\partial \Psi_{4l}^{*}}{\partial x_l}\, \Psi_{4k} \right\} + \\
&+ \bar{\bar{c}}{}_2''\, s_0\, c \left(\sum_{k=1}^{3} \Psi_{4k}^{*}\, \Psi_{5k} + \sum_{k=1}^{3} \Psi_{5k}^{*}\, \Psi_{4k} \right) - \qquad\qquad (378) \\
&- c_2''\, c \sum_{k=1}^{3} s_{1k}\, \frac{1}{\varkappa} \left\{ \sum_{l=1}^{3} \left(\frac{\partial \Psi_{5k}^{*}}{\partial x_l} - \frac{\partial \Psi_{5l}^{*}}{\partial x_k} \right) \Psi_{5l} + \sum_{l=1}^{3} \Psi_{5l}^{*} \left(\frac{\partial \Psi_{5k}}{\partial x_l} - \frac{\partial \Psi_{5l}}{\partial x_k} \right) + \right. \\
&\qquad\qquad \left. + \Psi_{4k}^{*} \sum_{l=1}^{3} \frac{\partial \Psi_{4l}}{\partial x_l} + \sum_{l=1}^{3} \frac{\partial \Psi_{4l}^{*}}{\partial x_l}\, \Psi_{4k} \right\} + \\
&+ i\, c_3'\, c \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{3kl} \left\{ \Psi_{4k}^{*}\, \Psi_{4l} - \frac{1}{\varkappa^2} \sum_{n=1}^{3} \left(\frac{\partial \Psi_{5n}^{*}}{\partial x_k} - \frac{\partial \Psi_{5k}^{*}}{\partial x_n} \right) \left(\frac{\partial \Psi_{5n}}{\partial x_l} - \frac{\partial \Psi_{5l}}{\partial x_n} \right) \right\} - \\
&- i\, c_3'\, c \sum_{k=1}^{3} \sum_{l=1}^{3} s_{2k}\, \frac{1}{\varkappa} \left\{ \Psi_{4l}^{*} \left(\frac{\partial \Psi_{5l}}{\partial x_k} - \frac{\partial \Psi_{5k}}{\partial x_l} \right) - \left(\frac{\partial \Psi_{5l}^{*}}{\partial x_k} - \frac{\partial \Psi_{5k}^{*}}{\partial x_l} \right) \Psi_{4l} \right\} + \\
&+ i\, \bar{\bar{c}}{}_3''\, c \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{3kl}\, \Psi_{5k}^{*}\, \Psi_{5l} -
\end{aligned}
$$

$$\begin{aligned}
&- i\,\bar{\bar{c}}_3''\, c \sum_{k=1}^{3} s_{2k} \frac{1}{\bar{\bar{\varkappa}}} \sum_{l=1}^{3} \left(\frac{\partial \Psi_{4l}^*}{\partial x_l} \Psi_{5k} - \Psi_{5k}^* \frac{\partial \Psi_{4l}}{\partial x_l} \right) + \\
&+ \bar{\bar{c}}_4'\, c \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{0kl} \Bigg\{ \Psi_{4k}^* \Psi_{5l} - \Psi_{5k}^* \Psi_{4l} - \\
&\qquad - \frac{1}{\bar{\bar{\varkappa}}^2} \frac{1}{2} \sum_{n=1}^{3} \frac{\partial \Psi_{4n}^*}{\partial x_n} \left(\frac{\partial \Psi_{5l}}{\partial x_k} - \frac{\partial \Psi_{5k}}{\partial x_l} \right) - \frac{1}{\bar{\bar{\varkappa}}^2} \frac{1}{2} \sum_{n=1}^{3} \left(\frac{\partial \Psi_{5l}^*}{\partial x_k} - \frac{\partial \Psi_{5k}^*}{\partial x_l} \right) \frac{\partial \Psi_{4n}}{\partial x_n} \Bigg\} + \\
&+ c_4'\, c\, s_1 \cdot \frac{1}{\varkappa} \left\{ \Psi_{52}^* \left(\frac{\partial \Psi_{51}}{\partial x_3} - \frac{\partial \Psi_{53}}{\partial x_1} \right) + \left(\frac{\partial \Psi_{51}^*}{\partial x_3} - \frac{\partial \Psi_{53}^*}{\partial x_1} \right) \Psi_{52} + \text{cycl. perm. of } 1, 2, 3 \right\} + \\
&+ i\,\bar{\bar{c}}_4''\, c \sum_{k=1}^{3}{}' \sum_{l=1}^{3}{}' s_{0kl} \Bigg\{ \Psi_{4k}^* \Psi_{5l} + \Psi_{5k}^* \Psi_{4l} + \\
&\qquad + \frac{1}{\bar{\bar{\varkappa}}^2} \frac{1}{2} \sum_{n=1}^{3} \frac{\partial \Psi_{4n}^*}{\partial x_n} \left(\frac{\partial \Psi_{5l}}{\partial x_k} - \frac{\partial \Psi_{5k}}{\partial x_l} \right) - \frac{1}{\bar{\bar{\varkappa}}^2} \frac{1}{2} \sum_{n=1}^{3} \left(\frac{\partial \Psi_{5l}^*}{\partial x_k} - \frac{\partial \Psi_{5k}^*}{\partial x_l} \right) \frac{\partial \Psi_{4n}}{\partial x_n} \Bigg\} - \\
&- i\,\bar{\bar{c}}_4''\, c\, s_1 \frac{1}{\bar{\bar{\varkappa}}} \left\{ \Psi_{52}^* \left(\frac{\partial \Psi_{51}}{\partial x_3} - \frac{\partial \Psi_{53}}{\partial x_1} \right) - \left(\frac{\partial \Psi_{51}^*}{\partial x_3} - \frac{\partial \Psi_{53}^*}{\partial x_1} \right) \Psi_{52} + \text{cycl. perm. of } 1, 2, 3 \right\} - \\
&- c_5\, c\, s_2 \frac{1}{\bar{\bar{\varkappa}}} \left\{ \Psi_{42}^* \left(\frac{\partial \Psi_{51}}{\partial x_3} - \frac{\partial \Psi_{53}}{\partial x_1} \right) + \left(\frac{\partial \Psi_{52}^*}{\partial x_1} - \frac{\partial \Psi_{51}^*}{\partial x_2} \right) \Psi_{43} + \text{cycl. perm. of } 1, 2, 3 \right\} \Bigg]\, d\,V.
\end{aligned}$$

We will now introduce the two vectors $\boldsymbol{U}$ and $\boldsymbol{P}$, with the components U_1, U_2, U_3 and P_1, P_2, P_3 defined by the equations

$$\Psi_{5k} = \frac{1}{\hbar} \sqrt{\frac{\bar{\bar{\mu}}}{8\pi}}\, U_k \qquad (k = 1, 2, 3) \tag{379}$$

and

$$\Psi_{4k} = \sqrt{\frac{8\pi}{\bar{\bar{\mu}}}}\, P_k^* \qquad (k = 1, 2, 3). \tag{380}$$

Since from equ. (278)

$$\Pi_{4k} = \sum_{m'=1}^{5} \sum_{n'=1}^{5} i\, \delta_{44}\, \delta_{m'5}\, \delta_{kn'}\, \Psi_{m'n'} = i\, \Psi_{5k} \tag{381}$$

equ. (280) gives the ordinary canonical commutation rule:

$$[U_k(\boldsymbol{r}'), P_k(\boldsymbol{r}'')] = i\hbar\, \delta(\boldsymbol{r}' - \boldsymbol{r}''). \tag{382}$$

With the dimensionless constants $\bar{\bar{C}}_k$, defined by

$$c_k = C_k \frac{\mathrm{e}^2}{\mu c^2} \frac{8\pi\hbar}{\varkappa}, \tag{383}$$

we finally obtain the Hamiltonian in the weak coupling approximation:

$$\begin{aligned}
H = \int \Big[& \frac{1}{8\pi} \operatorname{rot} \boldsymbol{U}^* \operatorname{rot} \boldsymbol{U} + 8\pi c^2 \boldsymbol{P} \boldsymbol{P}^* + \\
& \qquad + \frac{8\pi c^2}{\varkappa^2} \operatorname{div} \boldsymbol{P} \operatorname{div} \boldsymbol{P}^* + \frac{\varkappa^2}{8\pi} \boldsymbol{U}^* \boldsymbol{U} \Big] d V + \\
+ \frac{e^2}{\mu c^2} \int \Big[& \bar{\bar{C}}_1' s_3 \left\{ \boldsymbol{U}^* \boldsymbol{U} - \frac{1}{\varkappa^2} \left(\frac{8\pi c}{\varkappa} \right)^2 \operatorname{div} \boldsymbol{P} \operatorname{div} \boldsymbol{P}^* \right\} + \\
& + \bar{C}_1'' s_3 \left\{ \left(\frac{8\pi c}{\varkappa} \right)^2 \boldsymbol{P} \boldsymbol{P}^* - \frac{1}{\varkappa^2} \operatorname{rot} \boldsymbol{U}^* \operatorname{rot} \boldsymbol{U} \right\} + \\
& + \bar{C}_2' s_0 i \frac{8\pi c}{\varkappa} \{ \boldsymbol{P} \boldsymbol{U} - \boldsymbol{U}^* \boldsymbol{P}^* \} - \\
& - \bar{C}_2' i \frac{1}{\varkappa} \boldsymbol{s}_1 \left\{ \operatorname{rot} \boldsymbol{U}^* \times \boldsymbol{U} + \boldsymbol{U}^* \times \operatorname{rot} \boldsymbol{U} + \left(\frac{8\pi c}{\varkappa} \right)^2 (\boldsymbol{P} \operatorname{div} \boldsymbol{P}^* - \operatorname{div} \boldsymbol{P} \boldsymbol{P}^*) \right\} + \\
& + C_2'' s_0 \frac{8\pi c}{\varkappa} \{ \boldsymbol{P} \boldsymbol{U} + \boldsymbol{U}^* \boldsymbol{P}^* \} - \\
& - \bar{\bar{C}}_2'' \frac{1}{\varkappa} \boldsymbol{s}_1 \left\{ \operatorname{rot} \boldsymbol{U}^* \times \boldsymbol{U} - \boldsymbol{U}^* \times \operatorname{rot} \boldsymbol{U} + \left(\frac{8\pi c}{\varkappa} \right)^2 (\boldsymbol{P} \operatorname{div} \boldsymbol{P}^* + \operatorname{div} \boldsymbol{P} \boldsymbol{P}^*) \right\} + \\
& + C_3' i \boldsymbol{s}_3 \left\{ \left(\frac{8\pi c}{\varkappa} \right)^2 \boldsymbol{P} \times \boldsymbol{P}^* - \frac{1}{\varkappa^2} (\operatorname{rot} \boldsymbol{U}^* \times \operatorname{rot} \boldsymbol{U}) \right\} - \\
& - \bar{\bar{C}}_3' i \boldsymbol{s}_2 \frac{1}{\varkappa} \frac{8\pi c}{\varkappa} \{ \boldsymbol{P} \times \operatorname{rot} \boldsymbol{U} + \operatorname{rot} \boldsymbol{U}^* \times \boldsymbol{P}^* \} + \\
& + \bar{C}_3'' i \boldsymbol{s}_3 \{ \boldsymbol{U}^* \times \boldsymbol{U} \} - \qquad\qquad\qquad\qquad\qquad\qquad (384) \\
& - \bar{C}_3'' i \boldsymbol{s}_2 \frac{1}{\varkappa} \frac{8\pi c}{\varkappa} \{ \operatorname{div} \boldsymbol{P} \boldsymbol{U} - \boldsymbol{U}^* \operatorname{div} \boldsymbol{P}^* \} + \\
& + C_4' \boldsymbol{s}_0 \frac{8\pi c}{\varkappa} \left\{ \boldsymbol{P} \times \boldsymbol{U} - \boldsymbol{U}^* \times \boldsymbol{P}^* - \frac{1}{\varkappa^2} (\operatorname{div} \boldsymbol{P} \operatorname{rot} \boldsymbol{U} + \operatorname{rot} \boldsymbol{U}^* \operatorname{div} \boldsymbol{P}^*) \right\} + \\
& + \bar{\bar{C}}_4' s_1 \frac{1}{\varkappa} \{ \operatorname{rot} \boldsymbol{U}^* \boldsymbol{U} + \boldsymbol{U}^* \operatorname{rot} \boldsymbol{U} \} + \\
& + \bar{C}_4'' i \boldsymbol{s}_0 \frac{8\pi c}{\varkappa} \left\{ \boldsymbol{P} \times \boldsymbol{U} + \boldsymbol{U}^* \times \boldsymbol{P}^* - \frac{1}{\varkappa^2} (- \operatorname{div} \boldsymbol{P} \operatorname{rot} \boldsymbol{U} + \operatorname{rot} \boldsymbol{U}^* \operatorname{div} \boldsymbol{P}^*) \right\} + \\
& + C_4'' i s_1 \frac{1}{\varkappa} \{ \operatorname{rot} \boldsymbol{U}^* \boldsymbol{U} - \boldsymbol{U}^* \operatorname{rot} \boldsymbol{U} \} - \\
& - C_5 s_2 \frac{1}{\varkappa} \frac{8\pi c}{\varkappa} \{ \boldsymbol{P} \operatorname{rot} \boldsymbol{U} + \operatorname{rot} \boldsymbol{U}^* \boldsymbol{P}^* \} \Big] d V.
\end{aligned}$$

Here the C_3''-term has been treated by KLEIN [20] and the C_1'-term by PAULI & HU [27]. All the terms containing s_0:s and s_3:s, i. e. terms to be used in a non-relativistic treatment of the nucleons, have been listed by BRULIN and the author [5].

Appendix

In this appendix we shall consider the alternative method of developing the pair theory, starting from a real, LORENTZ invariant LAGRANGE function.

Let us first treat the scalar case, for which the unperturbed Lagrangian density is

$$L_0 = -\frac{1}{8\pi}\left\{\sum_{k=1}^{4}\frac{\partial U^*}{\partial x_k}\frac{\partial U}{\partial x_k} + \varkappa^2 U^* U\right\}. \tag{385}$$

To this function we add a linear combination of all the possible bilinear LORENTZ invariant expressions, which can be formed from the scalars U and U^*, the 4-vectors $\frac{\partial U}{\partial x_k}$ and $\frac{\partial U^*}{\partial x_k}$ and the tensors (59). This combination can be written:

$$\begin{aligned} L' = -\frac{e^2}{\bar{\mu} c^2}\Bigg[& -\bar{C}_1'\tilde{\psi}\psi U^* U - \\ & -\frac{1}{\varkappa^2}\bar{C}_1''\tilde{\psi}\psi\sum_{k=1}^{4}\frac{\partial U^*}{\partial x_k}\frac{\partial U}{\partial x_k} - \\ & -\frac{1}{\varkappa}\bar{C}_2'\sum_{k=1}^{4}\tilde{\psi}\gamma_k\psi\left(\frac{\partial U^*}{\partial x_k}U - U^*\frac{\partial U}{\partial x_k}\right) + \\ & + i\frac{1}{\varkappa}\bar{C}_2''\sum_{k=1}^{4}\tilde{\psi}\gamma_k\psi\left(\frac{\partial U^*}{\partial x_k}U + U^*\frac{\partial U}{\partial x_k}\right) - \\ & -\frac{1}{\varkappa^2}\bar{C}_3\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\tilde{\psi}\gamma_k\gamma_l\psi\frac{\partial U^*}{\partial x_k}\frac{\partial U}{\partial x_l}\Bigg]. \end{aligned} \tag{386}$$

With the Lagrangian density

$$L = L_0 + L'$$

LAGRANGE's equations of motion,

$$\sum_{k=1}^{4}\frac{\partial}{\partial x_k}\frac{\partial L}{\partial\frac{\partial U^*}{\partial x_k}} - \frac{\partial L}{\partial U^*} = 0, \tag{387}$$

give the following field equation for U:

$$\begin{aligned}
&\sum_{k=1}^{4} \frac{\partial^2 U}{\partial x_k^2} - \bar{\varkappa}^2 U = \\
&= -8\pi \frac{e^2}{\bar{\mu} c^2} \Big[\bar{C}_1' \tilde{\psi} \psi U - \\
&- \frac{1}{\bar{\varkappa}^2} \bar{C}_1'' \sum_{k=1}^{4} \frac{\partial}{\partial x_k} \left(\tilde{\psi} \psi \frac{\partial U}{\partial x_k} \right) - \\
&- \frac{1}{\bar{\varkappa}} \bar{C}_2' \sum_{k=1}^{4} \left\{ \frac{\partial}{\partial x_k} (\tilde{\psi} \gamma_k \psi U) + \tilde{\psi} \gamma_k \psi \frac{\partial U}{\partial x_k} \right\} + \\
&+ i \frac{1}{\bar{\varkappa}} \bar{C}_2'' \sum_{k=1}^{4} \left\{ \frac{\partial}{\partial x_k} (\tilde{\psi} \gamma_k \psi U) - \tilde{\psi} \gamma_k \psi \frac{\partial U}{\partial x_k} \right\} - \\
&- \frac{1}{\bar{\varkappa}^2} \bar{C}_3 \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \frac{\partial}{\partial x_k} \left(\tilde{\psi} \gamma_k \gamma_l \psi \frac{\partial U}{\partial x_l} \right) \Big]. \qquad (388)
\end{aligned}$$

This equation is, however, not exactly the same as the equation for Ψ_5, which results from eqs. (118) and (119) with the π_3 of equ. (331). On the contrary we get equ. (388) from equ. (118) if we put:

$$\begin{aligned}
\pi_3 = &-8\pi \frac{e^2}{\bar{\mu} c^2} \frac{\hbar^2}{i \bar{\mu} c} \Big[\bar{C}_1' \tilde{\psi} \psi \bar{\omega}_5 - \\
&- \frac{1}{\bar{\varkappa}^2} \bar{C}_1'' \sum_{k=1}^{4} \frac{\partial}{\partial x_k} \tilde{\psi} \psi \frac{\partial}{\partial x_k} \bar{\omega}_5 - \\
&- \frac{1}{\bar{\varkappa}} \bar{C}_2' \sum_{k=1}^{4} \left\{ \frac{\partial}{\partial x_k} \tilde{\psi} \gamma_k \psi + \tilde{\psi} \gamma_k \psi \frac{\partial}{\partial x_k} \right\} \bar{\omega}_5 + \\
&+ i \frac{1}{\bar{\varkappa}} \bar{C}_2'' \sum_{k=1}^{4} \left\{ \frac{\partial}{\partial x_k} \tilde{\psi} \gamma_k \psi - \tilde{\psi} \gamma_k \psi \frac{\partial}{\partial x_k} \right\} \bar{\omega}_5 - \\
&- \frac{1}{\bar{\varkappa}^2} \bar{C}_3 \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \frac{\partial}{\partial x_k} \tilde{\psi} \gamma_k \gamma_l \psi \frac{\partial}{\partial x_l} \bar{\omega}_5 \Big]. \qquad (389)
\end{aligned}$$

This $\bar{\pi}_3$ evidently conforms to eqs. (194) and (196), but not to equ. (198) since it contains derivatives (momentum and energy operators), which do not give the same result when they operate to the left or to the right. This means, that the 4-divergence of the 4-vector $\tilde{\Psi} \beta_k \Psi$ does not vanish, and consequently this 4-vector can no more represent the current 4-vector, if we want the meson charge to be conserved separately. Since, however, all terms in the Lagrangian density contain the meson field as an U^*-factor times an U-factor, the density is invariant for the transformation.

$$U \rightarrow U e^{i\alpha}, \qquad U^* \rightarrow U^* e^{-i\alpha}$$

where α is a real constant (gauge invariance of the first kind). This means, as is well-known, that the 4-divergence of the 4-vector

$$\frac{\partial \boldsymbol{L}}{\partial \frac{\partial \psi}{\partial x_k}} \psi - \psi^* \frac{\partial \boldsymbol{L}}{\partial \frac{\partial \psi^*}{\partial x_k}} \tag{390}$$

vanishes. This vector can consequently be chosen as the current 4-vector of the mesons, giving a pair theory in which the charge of the mesons is conserved separately. The fact that this 4-vector is not identical with the current vector $\tilde{\Psi} \beta_k \Psi$ of the DIRAC equation of the meson, seems to be a disadvantage of pair interactions of the form (389), which as we have seen have the greatest simplicity from the point of view of the LAGRANGE formalism.

In passing to the HAMILTON formalism we have to introduce the conjugate momenta, defined by

$$P = \frac{1}{i\,c} \frac{\partial \boldsymbol{L}}{\partial \frac{\partial U}{\partial x_4}} \quad \text{and} \quad P^* = \frac{1}{i\,c} \frac{\partial \boldsymbol{L}}{\partial \frac{\partial U^*}{\partial x_4}}. \tag{391}$$

Solving these equations with respect to the time derivatives and introducing them in the expression

$$\boldsymbol{H} = i\,c\,P \frac{\partial U}{\partial x_4} + i\,c \frac{\partial U^*}{\partial x_4} P^* - \boldsymbol{L}, \tag{392}$$

we obtain the Hamiltonian density as a function only of U, U^*, P, P^* and their spatial derivatives. Let us, as an example, write down the exact result of this procedure for the case, corresponding to that, for which we performed the exact elimination of the redundant components in Chapter VII, i. e. $\bar{C}_1'$ and $\bar{C}_1'' \neq 0$ and all other $\bar{C}_k$:s $= 0$. If we introduce the $\bar{c}_k$:s of equ. (356) and Ψ_4 and Ψ_5 by eqs. (351) and (352), we obtain as the additional term to the Hamiltonian density of the free space:

$$c\,\bar{c}_1'\,s_3\,\Psi_5^*\,\Psi_5 - \bar{\mu}\,c^2 \frac{\bar{c}_1''\,s_3}{\bar{\mu}\,c + \bar{c}_1''\,s_3}\,\Psi_4^*\,\Psi_4 + \frac{\hbar^2}{\bar{\mu}^2\,c}\,\bar{c}_1''\,s_3 \sum_{k=1}^{3} \frac{\partial \Psi_5^*}{\partial x_k} \frac{\partial \Psi_5}{\partial x_k}. \tag{393}$$

This density is clearly not identical with the integrand of the expression (345). However, in the weak coupling approximation, when $\bar{c}_1''\,s_3$ is small compared to $\bar{\mu}\,c$, the expressions are identical as regards terms of the first order in $\bar{c}_1''$, both giving the expression (346).

The identity of the two formalisms in first order of approximation is more general. In fact it is valid not only for the interactions $\bar{C}_1'$ and $\bar{C}_1''$, but for all the interactions of equ. (386). This can be proved in the following way:

Let us put

$$\boldsymbol{L} = -\frac{1}{8\pi}\left\{\sum_{k=1}^{4}\frac{\partial U^*}{\partial x_k}\frac{\partial U}{\partial x_k} + \varkappa^2 U^* U\right\} - \varepsilon f, \tag{394}$$

where ε is a small parameter and f is an arbitrary polynomial of U and U^* and their spatial and time derivatives. We have:

$$ic P = -\frac{1}{8\pi}\frac{\partial U^*}{\partial x_4} - \varepsilon\frac{\partial f}{\partial\dfrac{\partial U}{\partial x_4}}, \tag{395}$$

$$ic P^* = -\frac{1}{8\pi}\frac{\partial U}{\partial x_4} - \varepsilon\frac{\partial f}{\partial\dfrac{\partial U^*}{\partial x_4}}. \tag{396}$$

Only retaining up to linear terms in ε, we obtain:

$$\boldsymbol{H} = ic P\frac{\partial U}{\partial x_4} + ic\frac{\partial U^*}{\partial x_4}P^* - \boldsymbol{L} =$$

$$= -ic P\left(8\pi ic P^* + 8\pi\varepsilon\frac{\partial f}{\partial\dfrac{\partial U^*}{\partial x_4}}\right) - ic\left(8\pi ic P + 8\pi\varepsilon\frac{\partial f}{\partial\dfrac{\partial U}{\partial x_4}}\right)P^* +$$

$$+ \frac{1}{8\pi}\left(8\pi ic P + 8\pi\varepsilon\frac{\partial f}{\partial\dfrac{\partial U}{\partial x_k}}\right)\left(8\pi ic P^* + 8\pi\varepsilon\frac{\partial f}{\partial\dfrac{\partial U^*}{\partial x_4}}\right) +$$

$$+ \frac{1}{8\pi}\sum_{k=1}^{3}\frac{\partial U^*}{\partial x_k}\frac{\partial U}{\partial x_k} + \frac{\varkappa^2}{8\pi}U^* U + \varepsilon f =$$

$$= 8\pi c^2 P P^* + \frac{1}{8\pi}\sum_{k=1}^{3}\frac{\partial U^*}{\partial x_k}\frac{\partial U}{\partial x_k} + \frac{\varkappa^2}{8\pi}U^* U + \varepsilon f. \tag{397}$$

When we eliminate the time derivatives in the term εf of this result, we only need to use ε-independent approximations of the expressions for the time-derivatives, since the term εf already contains an ε. Consequently the additional term to the Hamiltonian is in first order of approximation equal to $-\boldsymbol{L}'$ of equ. (386), where we have made the substitutions

$$\frac{\partial U}{\partial x_4} = -8\pi ic P^* \tag{398}$$

and

$$\frac{\partial U^*}{\partial x_4} = -8\pi ic P. \tag{399}$$

Making use of eqs. (64)—(71) and (332)—(335) we obtain by this substitution precisely equ. (357), and the equivalence in first order of approximation with the DIRAC equation formalism is thus proved.

Let us now consider the vector case, for which the unperturbed Lagrangian density is

$$\boldsymbol{L}_0 = -\frac{1}{8\pi}\left\{\frac{1}{2}\sum_{k=1}^{4}\sum_{l=1}^{4}\left(\frac{\partial \tilde{U}_k}{\partial x_l}-\frac{\partial \tilde{U}_l}{\partial x_k}\right)\left(\frac{\partial U_k}{\partial x_l}-\frac{\partial U_l}{\partial x_k}\right)+\varkappa^2\sum_{k=1}^{4}\tilde{U}_k U_k\right\} \tag{400}$$

where the 4-vector $\tilde{U}_k$ is related to the 4-vector U_k by the equations

$$\tilde{U}_k = U_k^* \;\; (k=1,2,3) \qquad \text{and} \qquad \tilde{U}_4 = -U_4^*. \tag{401}$$

In order to obtain the interactions which correspond to, and, as we shall see, are in first order of approximation identical with the interaction (363), we have to add to $\boldsymbol{L}_0$ a linear combination of all the bilinear, LORENTZ invariant expressions, which can be formed from the vectors $\tilde{U}_k$ and U_k, the tensors $\left(\frac{\partial \tilde{U}_k}{\partial x_l}-\frac{\partial \tilde{U}_l}{\partial x_k}\right)$ and $\left(\frac{\partial U_k}{\partial x_l}-\frac{\partial U_l}{\partial x_k}\right)$ and the tensors (59). Since such interactions only contain antisymmetrical combinations of the derivatives, no time derivatives of $\tilde{U}_4$ and U_4 are introduced, so that these components have still no conjugate momenta and have thus still to be eliminated by passing to the HAMILTON formalism. We obtain:

$$\begin{aligned}
\boldsymbol{L}' = -\frac{e^2}{\bar{\bar{\mu}}c^2}\Bigg[&-\bar{\bar{C}}'_1\,\tilde{\psi}\psi\sum_{k=1}^{4}\tilde{U}_k U_k + \\
&+\frac{1}{\bar{\bar{\varkappa}}^2}\bar{\bar{C}}''_1\,\tilde{\psi}\psi\,\tfrac{1}{2}\sum_{k=1}^{4}\sum_{l=1}^{4}\left(\frac{\partial \tilde{U}_k}{\partial x_l}-\frac{\partial \tilde{U}_l}{\partial x_k}\right)\left(\frac{\partial U_k}{\partial x_l}-\frac{\partial U_l}{\partial x_k}\right)+ \\
&+\frac{1}{\bar{\bar{\varkappa}}}C'_2\sum_{k=1}^{4}\sum_{l=1}^{4}\tilde{\psi}\gamma_k\psi\left\{\left(\frac{\partial \tilde{U}_k}{\partial x_l}-\frac{\partial \tilde{U}_l}{\partial x_k}\right)U_l-\tilde{U}_l\left(\frac{\partial U_k}{\partial x_l}-\frac{\partial U_l}{\partial x_k}\right)\right\}- \\
&-i\frac{1}{\bar{\bar{\varkappa}}}\bar{\bar{C}}''_2\sum_{k=1}^{4}\sum_{l=1}^{4}\tilde{\psi}\gamma_k\psi\left\{\left(\frac{\partial \tilde{U}_k}{\partial x_l}-\frac{\partial \tilde{U}_l}{\partial x_k}\right)U_l+\tilde{U}_l\left(\frac{\partial U_k}{\partial x_l}-\frac{\partial U_l}{\partial x_k}\right)\right\}+ \\
&+\frac{1}{\bar{\bar{\varkappa}}^2}\bar{\bar{C}}'_3\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\sum_{m=1}^{4}\tilde{\psi}\gamma_k\gamma_l\psi\left(\frac{\partial \tilde{U}_k}{\partial x_m}-\frac{\partial \tilde{U}_m}{\partial x_k}\right)\left(\frac{\partial U_l}{\partial x_m}-\frac{\partial U_m}{\partial x_l}\right)- \\
&-\bar{C}''_3\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\tilde{\psi}\gamma_k\gamma_l\psi\,\tfrac{1}{2}\left(\tilde{U}_k U_l-\tilde{U}_l U_k\right)- \qquad\qquad (402) \\
&-\frac{1}{\varkappa}\bar{C}'_4\sum_{k=1}^{4}{}'\sum_{l=1}^{4}{}'\sum_{m=1}^{4}{}'\tilde{\psi}\gamma_k\gamma_l\gamma_m\psi\,\tfrac{1}{2}\left\{\left(\frac{\partial \tilde{U}_k}{\partial x_l}-\frac{\partial \tilde{U}_l}{\partial x_k}\right)U_m+\tilde{U}_m\left(\frac{\partial U_k}{\partial x_l}-\frac{\partial U_l}{\partial x_k}\right)\right\}-
\end{aligned}$$

$$- i \frac{1}{\bar{\bar{\varkappa}}} \bar{\bar{C}}_4'' \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \tilde{\psi} \gamma_k \gamma_l \gamma_m \psi \tfrac{1}{2} \left\{ \left(\frac{\partial \tilde{U}_k}{\partial x_l} - \frac{\partial \tilde{U}_l}{\partial x_k} \right) U_m - \tilde{U}_m \left(\frac{\partial U_k}{\partial x_l} - \frac{\partial U_l}{\partial x_k} \right) \right\} -$$

$$- \frac{1}{\varkappa^2} \bar{\bar{C}}_5 \sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \tilde{\psi} \gamma_k \gamma_l \gamma_m \gamma_n \psi \tfrac{1}{4} \left(\frac{\partial \tilde{U}_k}{\partial x_l} - \frac{\partial \tilde{U}_l}{\partial x_k} \right) \left(\frac{\partial U_m}{\partial x_n} - \frac{\partial U_n}{\partial x_m} \right) \Bigg] .$$

LAGRANGE's equations of motion now give with $\boldsymbol{L} = \boldsymbol{L}_0 + \boldsymbol{L}'$ for the components $s = 1, 2, 3, 4$:

$$\sum_{r=1}^{4} \frac{\partial}{\partial x_r} \left(\frac{\partial U_s}{\partial x_r} - \frac{\partial U_r}{\partial x_s} \right) - \bar{\bar{\varkappa}}^2 U_s =$$

$$= - 8 \pi \frac{e^2}{\bar{\mu} c^2} \Bigg[C_1' \tilde{\psi} \psi U_s +$$

$$+ \frac{1}{\bar{\bar{\varkappa}}^2} \bar{\bar{C}}_1'' \sum_{r=1}^{4} \frac{\partial}{\partial x_r} \tilde{\psi} \psi \left(\frac{\partial U_s}{\partial x_r} - \frac{\partial U_r}{\partial x_s} \right) +$$

$$+ \frac{1}{\varkappa} \bar{\bar{C}}_2' \sum_{r=1}^{4} \left\{ \frac{\partial}{\partial x_r} (\tilde{\psi} \gamma_s \psi U_r - \tilde{\psi} \gamma_r \psi U_s) + \tilde{\psi} \gamma_r \psi \left(\frac{\partial U_r}{\partial x_s} - \frac{\partial U_s}{\partial x_r} \right) \right\} -$$

$$- i \frac{1}{\varkappa} \bar{C}_2'' \sum_{r=1}^{4} \left\{ \frac{\partial}{\partial x_r} (\tilde{\psi} \gamma_s \psi U_r - \tilde{\psi} \gamma_r \psi U_s) - \tilde{\psi} \gamma_r \psi \left(\frac{\partial U_r}{\partial x_s} - \frac{\partial U_s}{\partial x_r} \right) \right\} +$$

$$+ \frac{1}{\bar{\bar{\varkappa}}^2} \bar{\bar{C}}_3' \sum_{r=1}^{4}{}' \sum_{l=1}^{4}{}^{\prime\,(l \neq s)} \frac{\partial}{\partial x_r} \left\{ \tilde{\psi} \gamma_s \gamma_l \psi \left(\frac{\partial U_l}{\partial x_r} - \frac{\partial U_r}{\partial x_l} \right) - \tilde{\psi} \gamma_r \gamma_l \psi \left(\frac{\partial U_l}{\partial x_s} - \frac{\partial U_s}{\partial x_l} \right) \right\} +$$

$$+ \bar{\bar{C}}_3'' \sum_{r=1}^{4}{}^{(r \neq s)} \tilde{\psi} \gamma_s \gamma_r \psi U_r -$$

$$- \frac{1}{\bar{\bar{\varkappa}}} \bar{\bar{C}}_4' \sum_{r=1}^{4}{}^{\prime\,(r \neq s)} \sum_{m=1}^{4}{}^{\prime\,(m \neq s)} \left\{ \frac{\partial}{\partial x_r} (\tilde{\psi} \gamma_s \gamma_r \gamma_m \psi U_m) - \tilde{\psi} \gamma_r \gamma_m \gamma_s \psi \frac{\partial U_r}{\partial x_m} \right\} -$$

$$- i \frac{1}{\bar{\bar{\varkappa}}} \bar{\bar{C}}_4'' \sum_{r=1}^{4}{}^{\prime\,(r \neq s)} \sum_{m=1}^{4}{}^{\prime\,(m \neq s)} \left\{ \frac{\partial}{\partial x_r} (\tilde{\psi} \gamma_s \gamma_r \gamma_m \psi U_m) + \tilde{\psi} \gamma_r \gamma_m \gamma_s \psi \frac{\partial U_r}{\partial x_m} \right\} -$$

$$- \frac{1}{\bar{\bar{\varkappa}}^2} \bar{C}_5 \sum_{r=1}^{4}{}^{\prime\,(r \neq s)} \sum_{m=1}^{4}{}^{\prime\,(m \neq s)} \sum_{n=1}^{4}{}^{\prime\,(n \neq s)} \frac{\partial}{\partial x_r} \left(\tilde{\psi} \gamma_s \gamma_r \gamma_m \gamma_n \psi \frac{\partial U_m}{\partial x_n} \right) \Bigg] . \qquad (403)$$

The same field equations can be obtained from the DIRAC formalism with a suitable choice of $\bar{\bar{\pi}}_\beta$. In fact, we get by eqs. (170) and (172)

$$\sum_{k=1}^{4} \frac{\partial}{\partial x_k} \left(\frac{\partial \Psi_{5s}}{\partial x_k} - \frac{\partial \Psi_{5k}}{\partial x_s} \right) - \bar{\bar{\varkappa}}^2 \Psi_{5s} =$$

$$= \frac{1}{\hbar} \sum_{k=1}^{4} \frac{\partial}{\partial x_k} (\bar{\bar{\pi}}_\beta \Psi)_{ks} + \frac{i \bar{\mu} c}{\hbar^2} (\bar{\bar{\pi}}_\beta \Psi)_{5s} \qquad (404)$$

and these equations give exactly eqs. (403) if we put

$$\begin{aligned}\bar{\bar{\pi}}_3 = -8\pi \frac{e^2}{\bar{\mu} c^2} \frac{\hbar^2}{i \bar{\bar{\mu}} c} \Big[& \bar{\bar{C}}_1' \tilde{\psi} \psi \bar{\omega}_5 + \\ & + \frac{1}{\bar{\bar{\varkappa}}^2} \bar{C}_1'' \sum_{k=1}^{4} \sum_{l=1}^{4} \frac{\partial}{\partial x_k} \bar{\beta}_k \tilde{\psi} \psi \beta_l \frac{\partial}{\partial x_l} \bar{\bar{\omega}}_5 - \\ & - \frac{1}{\varkappa} C_2' \sum_{k=1}^{4} \sum_{l=1}^{4} \left(\frac{\partial}{\partial x_k} \beta_k \beta_l \tilde{\psi} \gamma_l \psi + \tilde{\psi} \gamma_l \psi \beta_l \beta_k \frac{\partial}{\partial x_k} \right) \omega_5 + \\ & + i \frac{1}{\varkappa} C_2'' \sum_{k=1}^{4} \sum_{l=1}^{4} \left(\frac{\partial}{\partial x_k} \beta_k \bar{\beta}_l \tilde{\psi} \gamma_l \psi - \tilde{\psi} \gamma_l \psi \beta_l \beta_k \frac{\partial}{\partial x_k} \right) \omega_5 + \\ & + \frac{1}{\varkappa^2} C_3' \sum_{k=1}^{4} {\sum_{l=1}^{4}}' {\sum_{m=1}^{4}}' \sum_{n=1}^{4} \frac{\partial}{\partial x_k} \beta_k \beta_l \tilde{\psi} \gamma_l \gamma_m \psi \beta_m \beta_n \frac{\partial}{\partial x_n} \omega_5 + \\ & + C_3'' {\sum_{k=1}^{4}}' {\sum_{l=1}^{4}}' \beta_k \tilde{\psi} \gamma_k \gamma_l \psi \beta_l \omega_5 - \\ & - \frac{1}{\varkappa} \bar{C}_4' {\sum_{k=1}^{4}}' {\sum_{l=1}^{4}}' {\sum_{m=1}^{4}}' \left(\frac{\partial}{\partial x_m} \bar{\bar{\beta}}_k \tilde{\psi} \gamma_k \gamma_m \gamma_l \psi \bar{\bar{\beta}}_l + \bar{\bar{\beta}}_l \tilde{\psi} \gamma_l \gamma_m \gamma_k \psi \bar{\bar{\beta}}_k \frac{\partial}{\partial x_m} \right) \omega_5 - \\ & - i \frac{1}{\varkappa} \bar{\bar{C}}_4'' {\sum_{k=1}^{4}}' {\sum_{l=1}^{4}}' {\sum_{m=1}^{4}}' \left(\frac{\partial}{\partial x_m} \bar{\bar{\beta}}_k \tilde{\psi} \gamma_k \gamma_m \gamma_l \psi \bar{\bar{\beta}}_l - \bar{\bar{\beta}}_l \tilde{\psi} \gamma_l \gamma_m \gamma_k \psi \bar{\bar{\beta}}_k \frac{\partial}{\partial x_m} \right) \bar{\bar{\omega}}_5 + \\ & + \frac{1}{\varkappa^2} \bar{\bar{C}}_5 {\sum_{k=1}^{4}}' {\sum_{l=1}^{4}}' {\sum_{m=1}^{4}}' {\sum_{n=1}^{4}}' \frac{\partial}{\partial x_k} \bar{\bar{\beta}}_l \tilde{\psi} \gamma_k \gamma_l \gamma_m \gamma_n \psi \bar{\bar{\beta}}_m \frac{\partial}{\partial x_n} \bar{\omega}_5 \Big], \end{aligned} \tag{405}$$

and make use of the fact that

$$(\bar{\beta}_k \bar{\bar{\beta}}_l \Psi)_{5s} = \delta_{kl} \Psi_{5s} - \delta_{sl} \Psi_{5k}, \tag{406}$$

which can easily be proved by means of equ. (139). Just as in the scalar case a 4-vector can be constructed, the 4-divergence of which vanishes, but this 4-vector is no more the DIRAC current 4-vector $\tilde{\Psi} \beta_k \Psi$, as was the case with the interaction (363).

In this case too, the two interactions (402) and (363) give in first order of approximation the same Hamiltonian. To prove this, let us put

$$\boldsymbol{L} = \boldsymbol{L}_0 - \varepsilon f, \tag{407}$$

where ε is a small parameter and f is an arbitrary polynomial of U_k, $\tilde{U}_k$ and their derivatives, not containing time derivatives of U_4 and $\tilde{U}_4$.

This Lagrangian density gives:

$$\tilde{P}_r = \frac{1}{ic} \frac{\partial \boldsymbol{L}}{\partial \frac{\partial \tilde{U}_r}{\partial x_4}} = -\frac{1}{8\pi i c} \left(\frac{\partial U_r}{\partial x_4} - \frac{\partial U_4}{\partial x_r} \right) - \frac{1}{ic} \varepsilon \frac{\partial f}{\partial \frac{\partial \tilde{U}_r}{\partial x_4}} \quad (r = 1, 2, 3) \tag{408}$$

and

$$P_r = \frac{1}{ic}\,\frac{\partial L}{\partial \dfrac{\partial U_r}{\partial x_4}} = -\frac{1}{8\pi i c}\left(\frac{\partial \tilde U_r}{\partial x_4} - \frac{\partial \tilde U_4}{\partial x_r}\right) - \frac{1}{ic}\,\varepsilon\,\frac{\partial f}{\partial \dfrac{\partial U_r}{\partial x_4}} \quad (r = 1, 2, 3). \qquad (409)$$

We now obtain:

$$\begin{aligned} \boldsymbol{H} &= ic \sum_{r=1}^{3}\left(P_r \frac{\partial U_r}{\partial x_4} + \frac{\partial \tilde U_r}{\partial x_4}\tilde P_r\right) - \boldsymbol{L} = \\ &= ic \sum_{r=1}^{3}\left\{P_r\left(-8\pi i c \tilde P_r + \frac{\partial U_4}{\partial x_r} - 8\pi\varepsilon \frac{\partial f}{\partial \dfrac{\partial \tilde U_r}{\partial x_4}}\right) + \right. \\ &\quad \left. + \left(-8\pi i c P_r + \frac{\partial \tilde U_4}{\partial x_r} - 8\pi\varepsilon\frac{\partial f}{\partial \dfrac{\partial U_r}{\partial x_4}}\right)\tilde P_r\right\} + \qquad (410) \\ &\quad + \frac{1}{8\pi}\sum_{k=1}^{3}\left(\frac{\partial \tilde U_k}{\partial x_4} - \frac{\partial \tilde U_4}{\partial x_k}\right)\left(\frac{\partial U_k}{\partial x_4} - \frac{\partial U_4}{\partial x_k}\right) + \\ &\quad + \frac{1}{8\pi}\frac{1}{2}\sum_{k=1}^{3}\sum_{l=1}^{3}\left(\frac{\partial \tilde U_k}{\partial x_l} - \frac{\partial \tilde U_l}{\partial x_k}\right)\left(\frac{\partial U_k}{\partial x_l} - \frac{\partial U_l}{\partial x_k}\right) + \\ &\quad + \frac{\overset{=}{\varkappa}^2}{8\pi}\sum_{k=1}^{3}\tilde U_k U_k + \frac{\overset{=}{\varkappa}^2}{8\pi}\tilde U_4 U_4 + \varepsilon f. \end{aligned}$$

This gives, if we substitute $\left(\frac{\partial U_k}{\partial x_4} - \frac{\partial U_4}{\partial x_k}\right)$ and $\left(\frac{\partial \tilde U_k}{\partial x_4} - \frac{\partial \tilde U_4}{\partial x_k}\right)$ by means of eqs. (408) and (409) and only retain terms which are at most of the first degree in ε:

$$\begin{aligned} \boldsymbol{H} &= \frac{1}{8\pi}\operatorname{rot}\tilde{\boldsymbol{U}}\operatorname{rot}\boldsymbol{U} + 8\pi c^2 \boldsymbol{P}\tilde{\boldsymbol{P}} + \frac{\overset{=}{\varkappa}^2}{8\pi}\tilde{\boldsymbol{U}}\boldsymbol{U} + \\ &\quad + \frac{\overset{=}{\varkappa}^2}{8\pi}\tilde U_4 U_4 + ic\sum_{r=1}^{3}\left(P_r\frac{\partial U_4}{\partial x_r} + \frac{\partial \tilde U_4}{\partial x_r}\tilde P_r\right) + \varepsilon f. \qquad (411) \end{aligned}$$

By equ. (403) for $s = 4$ and its conjugate complex and by the definitions (408) and (409) of $\tilde P_r$ and P_r we obtain in first approximation:

$$U_4 = \frac{8\pi i c}{\overset{=}{\varkappa}^2}\operatorname{div}\tilde{\boldsymbol{P}} + \varepsilon g, \qquad (412)$$

$$\tilde U_4 = \frac{8\pi i c}{\overset{=}{\varkappa}^2}\operatorname{div}\boldsymbol{P} + \varepsilon h, \qquad (413)$$

where g and h are functions of $\tilde U_k$, U_k and their derivatives.

Inserting these values in equ. (411) and neglecting the quantity

$$\sum_{r=1}^{3} \frac{\partial}{\partial x_r} (P_r U_4 + \tilde{U}_4 P_r)$$

we obtain finally:

$$\boldsymbol{H} = \frac{1}{8\pi} \operatorname{rot} \tilde{\boldsymbol{U}} \operatorname{rot} \boldsymbol{U} + 8\pi c^2 \boldsymbol{P} \tilde{\boldsymbol{P}} + \frac{\bar{\bar{\varkappa}}^2}{8\pi} \tilde{\boldsymbol{U}} \boldsymbol{U} + \\ + \frac{8\pi c^2}{\bar{\bar{\varkappa}}^2} \operatorname{div} \boldsymbol{P} \operatorname{div} \tilde{\boldsymbol{P}} + \varepsilon f. \tag{414}$$

The additional term to the Hamiltonian is thus in first order of approximation obtained as the additional term to the Lagrangian with the sign reversed and with $\frac{\partial U_r}{\partial x_4} - \frac{\partial U_4}{\partial x_r}$, $\frac{\partial \tilde{U}_r}{\partial x_4} - \frac{\partial \tilde{U}_4}{\partial x_r}$, $\tilde{U}_4$ and U_4 substituted by means of the equations:

$$\frac{\partial U_r}{\partial x_4} - \frac{\partial U_4}{\partial x_r} = -8\pi i c \tilde{P}_r, \tag{415}$$

$$\frac{\partial \tilde{U}_r}{\partial x_4} - \frac{\partial \tilde{U}_4}{\partial x_r} = -8\pi i c P_r, \tag{416}$$

$$U_4 = \frac{8\pi i c}{\bar{\bar{\varkappa}}^2} \operatorname{div} \tilde{\boldsymbol{P}}, \tag{417}$$

$$\tilde{U}_4 = \frac{8\pi i c}{\bar{\bar{\varkappa}}^2} \operatorname{div} \boldsymbol{P}, \tag{418}$$

giving exactly equ. (384), which proves the equivalence.

Finally we want to draw attention to the fact, that if we no more reject the possibility of a conjugate momentum to U_4, we are not restricted to use only antisymmetrical combinations of the derivatives in the additional term to the Lagrangian. The possible terms are then the bilinear invariants, which can be formed from $\tilde{U}_k$, U_k, $\frac{\partial \tilde{U}_k}{\partial x_l}$, $\frac{\partial U_k}{\partial x_l}$ and the tensors (59). These terms are:

$$\tilde{\psi} \psi \sum_{k=1}^{4} \tilde{U}_k U_k, \tag{419}$$

$$\tilde{\psi} \psi \sum_{k=1}^{4} \sum_{l=1}^{4} \frac{\partial \tilde{U}_k}{\partial x_k} \frac{\partial U_l}{\partial x_l}, \tag{420}$$

$$\tilde{\psi} \psi \sum_{k=1}^{4} \sum_{l=1}^{4} \frac{\partial \tilde{U}_k}{\partial x_l} \frac{\partial U_k}{\partial x_l}, \tag{421}$$

$$\tilde{\psi}\psi \sum_{k=1}^{4} \sum_{l=1}^{4} \frac{\partial \tilde{U}_k}{\partial x_l} \frac{\partial U_l}{\partial x_k}, \tag{422}$$

$$\sum_{k=1}^{4} \sum_{l=1}^{4} \tilde{\psi} \gamma_k \psi \left(\frac{\partial \tilde{U}_l}{\partial x_l} U_k \pm \tilde{U}_k \frac{\partial U_l}{\partial x_l} \right), \tag{423}$$

$$\sum_{k=1}^{4} \sum_{l=1}^{4} \tilde{\psi} \gamma_k \psi \left(\frac{\partial \tilde{U}_l}{\partial x_k} U_l \pm \tilde{U}_l \frac{\partial U_l}{\partial x_k} \right), \tag{424}$$

$$\sum_{k=1}^{4} \sum_{l=1}^{4} \tilde{\psi} \gamma_k \psi \left(\frac{\partial \tilde{U}_k}{\partial x_l} U_l \pm \tilde{U}_l \frac{\partial U_k}{\partial x_l} \right), \tag{425}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \tilde{\psi} \gamma_k \gamma_l \psi \tilde{U}_k U_l, \tag{426}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4} \tilde{\psi} \gamma_k \gamma_l \psi \left(\frac{\partial \tilde{U}_k}{\partial x_m} \frac{\partial U_m}{\partial x_l} \pm \frac{\partial \tilde{U}_m}{\partial x_l} \frac{\partial U_k}{\partial x_m} \right), \tag{427}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4} \tilde{\psi} \gamma_k \gamma_l \psi \left(\frac{\partial \tilde{U}_k}{\partial x_l} \frac{\partial U_m}{\partial x_m} \pm \frac{\partial \tilde{U}_m}{\partial x_m} \frac{\partial U_k}{\partial x_l} \right), \tag{428}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4} \tilde{\psi} \gamma_k \gamma_l \psi \frac{\partial \tilde{U}_m}{\partial x_k} \frac{\partial U_m}{\partial x_l}, \tag{429}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4} \tilde{\psi} \gamma_k \gamma_l \psi \frac{\partial \tilde{U}_k}{\partial x_m} \frac{\partial U_l}{\partial x_m}, \tag{430}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \tilde{\psi} \gamma_k \gamma_l \gamma_m \psi \left(\frac{\partial \tilde{U}_k}{\partial x_l} U_m \pm \tilde{U}_m \frac{\partial U_k}{\partial x_l} \right), \tag{431}$$

$$\sum_{k=1}^{4}{}' \sum_{l=1}^{4}{}' \sum_{m=1}^{4}{}' \sum_{n=1}^{4}{}' \tilde{\psi} \gamma_k \gamma_l \gamma_m \gamma_n \psi \frac{\partial \tilde{U}_k}{\partial x_l} \frac{\partial U_m}{\partial x_n}, \tag{432}$$

a special linear combination of which of course gives the expression (402).

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